1. (a) A FUNCTION $f: X \to Y$ IS A RULE THAT ASSIGNS TO EACH $x \in X$ EXACTLY ONE $y \in Y$ (DENOTED $f(x)$)

X IS CALLED THE DOMAIN OF f (FROM), AND Y IS CALLED THE CODOMAIN OF f (TO).

(b) THE PRECISE LOGICAL DEFINITION IS:

$$\forall x \in X \ \underbrace{\exists!}_{\text{"THERE EXISTS UNIQUE"}} \ y \in Y \text{ WITH } f(x)=y$$

2. IF $f: X \to Y$, THE RANGE OF f IS THE SET OF ALL ITS OUTPUT VALUES, I.E.,

$$\text{RANGE } f = \{ f(x) : x \in X \}$$

(OR MORE RIGIDLY, $\{ y \in Y : \exists\, x \in X \text{ WITH } y = f(x) \}$)

RANGE(f) IS A SUBSET OF CODOMAIN(f); f HITS EACH ELEMENT OF ITS RANGE, BUT IT NEED NOT HIT EVERY ELEMENT OF ITS CODOMAIN.

3. (a) $f: X \to Y$ IS INJECTIVE IF $(\forall x_1, x_2 \in X)\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

I.E., $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$: NO TWO DISTINCT ELEMENTS OF THE DOMAIN ARE THROWN TO THE SAME ELEMENT OF THE CODOMAIN.

(b) $f: X \to Y$ IS SURJECTIVE IF $\forall y \in Y\ \exists x \in X$ WITH $f(x) = y$

I.E., EVERY ELEMENT y IN THE CODOMAIN IS HIT BY SOME ELEMENT x IN THE DOMAIN.

(c) f IS BIJECTIVE IF f IS BOTH INJECTIVE AND SURJECTIVE.

I.E., f GIVES A PAIRING BETWEEN THE ELEMENTS OF ITS DOMAIN AND THOSE OF ITS CODOMAIN.

4. GIVEN $f: X \to Y$ AND $g: Y \to Z$, THEIR COMPOSITION IS

$$g \circ f : X \to Z, \quad x \mapsto g(f(x))$$

I.E., $(g \circ f)(x) \overset{\text{DEF}}{=} g(f(x))$

5. IF $f_1, f_2 : X \to Y$, WE SAY THAT $f_1 = f_2$ IF $\forall x \in X,\ f_1(x) = f_2(x)$

I.E., FUNCTIONS ARE EQUAL $\Leftrightarrow$ THEY TAKE THE SAME VALUE FOR EACH ELEMENT IN THEIR DOMAIN.

6. FOR ANY SET X, THE IDENTITY FUNCTION ON X IS

$$id_X : X \to X, \quad x \mapsto x$$

I.E., $id_X(x) = x$ — id_X SENDS EACH $x \in X$ TO ITSELF.

(a) CLAIM: id_X IS BIJECTIVE

PROOF: TO SHOW id_X IS INJECTIVE,

(I.E., $\forall x_1, x_2 \in X$, $id_X(x_1) = id_X(x_2) \Rightarrow x_1 = x_2$)

LET $x_1, x_2 \in X$ BE GIVEN, AND SUPPOSE $id_X(x_1) = id_X(x_2)$.

THEN BY DEFINITION OF id_X,

$id_X(x_1) = x_1$ AND $id_X(x_2) = x_2$, SO $x_1 = x_2$ ✓

TO SHOW id_X IS SURJECTIVE,

(I.E., $\forall x \in X\ \exists x \in X$ WITH $id_X(x) = x$)

LET $x \in X$ BE GIVEN; TAKING THIS $x \in X$,

BY DEFINITION OF id_X, $id_X(x) = x$ ✓

id_X IS BOTH INJECTIVE + SURJECTIVE,

SO BY DEFINITION, id_X IS BIJECTIVE. ∎

(b) CLAIM: IF $f: X \to Y$, THEN $f \circ id_X = f$

PROOF: SUPPOSE $f: X \to Y$ IS A FUNCTION.

[GOAL: $f \circ id_X = f$, I.E., $\forall x \in X$, $(f \circ id_X)(x) = f(x)$]

LET $x \in X$ BE GIVEN.

THEN $(f \circ id_X)(x) \overset{\text{DEF}}{=} f(id_X(x)) \overset{\text{DEF}}{=} f(x)$ ∎

(c) CLAIM: IF $f: X \to Y$, THEN $id_Y \circ f = f$

PROOF: SUPPOSE $f: X \to Y$ IS A FUNCTION.

[GOAL: $id_Y \circ f = f$, I.E., $\forall x \in X$, $(id_Y \circ f)(x) = f(x)$]

LET $x \in X$ BE GIVEN.

THEN $(id_Y \circ f)(x) \overset{\text{DEF}}{=} id_Y(f(x)) \overset{\text{DEF}}{=} f(x)$ ∎

7. AN **INVERSE** FOR $f: X \to Y$ IS A FUNCTION $f^{-1}: Y \to X$ FOR WHICH

$$f^{-1} \circ f = id_X \quad \text{AND} \quad f \circ f^{-1} = id_Y$$

(I.E., $\forall x \in X,\ f^{-1}(f(x)) = x$ -AND- $\forall y \in Y,\ f(f^{-1}(y)) = y$)

f IS **INVERTIBLE** IF f HAS AN INVERSE.

(I.E., $\exists\, f^{-1}: Y \to X$ WITH $f^{-1} \circ f = id_X$ AND $f \circ f^{-1} = id_Y$)

8. $f: X \to Y$; $g: Y \to Z$

(a) INJECTIVITY + COMPOSITIONS:

(i) **CLAIM**: IF f IS INJECTIVE AND g IS INJECTIVE, THEN $g \circ f$ IS INJECTIVE

PROOF: SUPPOSE THAT f AND g ARE INJECTIVE,

I.E., $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ ①

AND $\forall y_1, y_2 \in Y$, $f(y_1) = f(y_2) \Rightarrow y_1 = y_2$ ②

[GOAL: $g \circ f$ IS INJECTIVE,

I.E., $\forall x_1, x_2 \in X$, $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$

LET $x_1, x_2 \in X$ BE GIVEN,

AND SUPPOSE THAT $(g \circ f)(x_1) = (g \circ f)(x_2)$.

THEN BY DEFINITION OF COMPOSITION, $g(\underbrace{f(x_1)}_{y_1}) = g(\underbrace{f(x_2)}_{y_2})$,

SO BY HYPOTHESIS ②, $f(x_1) = f(x_2)$

THUS BY HYPOTHESIS ①, $x_1 = x_2$.

SO, BY DEFINITION, $g \circ f$ IS INJECTIVE. ■

(ii) **CLAIM**: IF $g \circ f$ IS INJECTIVE, THEN f IS INJECTIVE.

PROOF: SUPPOSE $g \circ f$ IS INJECTIVE,

I.E., $\forall x_1, x_2 \in X$, $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$

[GOAL: f IS INJECTIVE, I.E., $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

LET $x_1, x_2 \in X$ BE GIVEN,

AND SUPPOSE $f(x_1) = f(x_2)$.

LOOK AT HYPOTHESIS: WE HAVE $x_1, x_2 \in X$; JUST APPLY g TO BOTH SIDES HERE

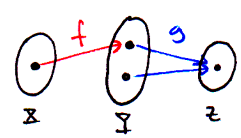
APPLYING g, $g(f(x_1)) = g(f(x_2))$,

I.E., $(g \circ f)(x_1) = (g \circ f)(x_2)$

SO BY HYPOTHESIS, $x_1 = x_2$.

THUS, BY DEFINITION, f IS INJECTIVE. ■

- g NEEDN'T BE INJECTIVE, BECAUSE THE RANGE OF f COULD "MISS" THE NON-INJECTIVITY OF g, E.G.:



$g \circ f$ IS INJECTIVE, EVEN THOUGH g IS NOT!

(b) SURJECTIVITY + COMPOSITIONS:

(i) <u>CLAIM</u>: IF f IS SURJECTIVE AND g IS SURJECTIVE, THEN $g \circ f$ IS SURJECTIVE.

<u>PROOF</u>: SUPPOSE THAT f AND g ARE SURJECTIVE,

I.E., $\forall\, y \in Y\ \exists\, x \in X$ WITH $f(x) = y$ ①

AND $\forall\, z \in Z\ \exists\, y \in Y$ WITH $g(y) = z$. ②

[<u>GOAL</u>: $g \circ f$ IS SURJECTIVE,
I.E., $\forall\, z \in Z\ \exists\, x \in X$ WITH $(g \circ f)(x) = z$

LET $z \in Z$ BE GIVEN. [<u>TO FIND AN $x \in X$</u>: WHAT WILL OUR z BUY US? ② LETS US BUY A $y \in Y$...

BY ②, $\exists\, y \in Y$ WITH $g(y) = z$.* [① NOW LETS US BUY AN $x \in X$

TAKING SUCH A $y \in Y$, BY ①, $\exists\, x \in X$ WITH $f(x) = y$.**

TAKE SUCH AN $x \in X$.

THEN $(g \circ f)(x) \overset{\text{DEF}}{=} g(f(x)) = g(y)$ FROM **

$= z$ FROM *.

THUS, BY DEFINITION, $g \circ f$ IS SURJECTIVE. ■

(ii) <u>CLAIM</u>: IF $g \circ f$ IS SURJECTIVE, THEN g IS SURJECTIVE.

<u>PROOF</u>: SUPPOSE THAT $g \circ f$ IS SURJECTIVE,

I.E., $\forall\, z \in Z\ \exists\, x \in X$ WITH $(g \circ f)(x) = z$. ①

[<u>GOAL</u>: SHOW g IS SURJECTIVE,
I.E., $\forall\, z \in Z\ \exists\, y \in Y$ WITH $g(y) = z$

LET $z \in Z$ BE GIVEN. [USE ① TO BUY AN $x \in X$!

THEN BY ①, $\exists\, x \in X$ WITH $(g \circ f)(x) = z$

TAKE SUCH AN $x \in X$; THEN $(g \circ f)(x) = z$,

I.E., $g(f(x)) = z$. *

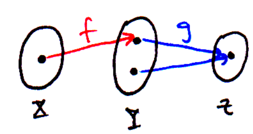
[WE NEED TO FIND A $y \in Y$ WITH $g(y) = z$ — TAKE $y = f(x)$

TAKE $y = f(x) \in Y$.

THEN $g(y) = g(f(x)) = z$ FROM *.

THUS, BY DEFINITION, g IS SURJECTIVE. ■

- f NEEDN'T BE SURJECTIVE, BECAUSE g COULD POSSIBLY STILL COVER Z FROM ONLY RANGE f, E.G.:



$g \circ f$ IS SURJECTIVE, EVEN THOUGH f IS NOT!

(c) BIJECTIVITY + COMPOSITIONS:

(i) CLAIM: IF f IS BIJECTIVE AND g IS BIJECTIVE, THEN $g \circ f$ IS BIJECTIVE.

PROOF: SUPPOSE f AND g ARE BIJECTIVE,
I.E., f IS INJECTIVE + SURJECTIVE
AND g IS INJECTIVE + SURJECTIVE

BY PART (a)(i), $f + g$ INJECTIVE $\Rightarrow g \circ f$ IS INJECTIVE;
BY PART (b)(i), $f + g$ SURJECTIVE $\Rightarrow g \circ f$ IS SURJECTIVE.
SO, BY DEFINITION, $g \circ f$ IS BIJECTIVE. ■

(ii) CLAIM: IF $g \circ f$ IS BIJECTIVE, THEN f IS INJECTIVE AND g IS SURJECTIVE.

PROOF: SUPPOSE $g \circ f$ IS BIJECTIVE,
I.E., $g \circ f$ IS INJECTIVE AND SURJECTIVE.

BY PART (a)(ii), $g \circ f$ INJECTIVE $\Rightarrow f$ IS INJECTIVE;
BY PART (b)(ii), $g \circ f$ SURJECTIVE $\Rightarrow g$ IS SURJECTIVE. ■

(d) (i) CLAIM: IF $f: X \to Y$ IS INVERTIBLE, THEN f AND f^{-1} ARE BIJECTIVE.

PROOF: SUPPOSE THAT $f: X \to Y$ IS INVERTIBLE,
I.E., f HAS AN INVERSE FUNCTION $f^{-1}: Y \to X$.
BY DEFINITION, $f^{-1} \circ f = id_X$ AND $f \circ f^{-1} = id_Y$.

BUT id_X IS BIJECTIVE, SO BY (c)(ii),
f IS INJECTIVE + f^{-1} IS SURJECTIVE.

SIMILARLY, id_Y IS BIJECTIVE, SO BY (c)(ii) AGAIN,
f^{-1} IS INJECTIVE + f IS SURJECTIVE.

SO, BY DEFINITION, f AND f^{-1} ARE BIJECTIVE. ■

(ii) CLAIM: IF f AND g ARE INVERTIBLE, THEN $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

PROOF: SUPPOSE f AND g ARE INVERTIBLE,
I.E., f HAS AN INVERSE FUNCTION f^{-1}
AND g HAS AN INVERSE FUNCTION g^{-1}.
BY DEFINITION OF INVERSE FUNCTION,
$f^{-1} \circ f = id_X$, $f \circ f^{-1} = id_Y$,
$g^{-1} \circ g = id_Y$, AND $g \circ g^{-1} = id_Z$.

GOAL: TO SHOW THAT $f^{-1} \circ g^{-1}$ IS THE INVERSE OF $g \circ f$,
I.E., THAT $(g \circ f) \circ (f^{-1} \circ g^{-1}) = id_Z$ AND $(f^{-1} \circ g^{-1}) \circ (g \circ f) = id_X$

WELL, $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1}$
$= g \circ id_Y \circ g^{-1}$ FROM ABOVE
$= g \circ g^{-1}$ BECAUSE id_Y IS THE IDENTITY
$= id_Z$ FROM ABOVE ✓

AND $(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f$
$= f^{-1} \circ id_Y \circ f$ FROM ABOVE
$= f^{-1} \circ f$ BECAUSE id_Y IS THE IDENTITY
$= id_X$ FROM ABOVE ✓

THUS, BY DEFINITION, $f^{-1} \circ g^{-1}$ IS THE INVERSE OF $g \circ f$ ■