1. (a) AN <u>ISOMORPHISM</u> BETWEEN VECTOR SPACES V AND W IS A <u>BIJECTIVE</u> FUNCTION $\phi: V \to W$ THAT <u>RESPECTS</u> <u>LINEAR COMBINATIONS</u>, I.E.,

$$\forall \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ AND SCALARS } a_1, a_2, \ldots, a_n,$$

$$\phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n)$$

* NOTE THAT THIS FORCES $\phi^{-1}: W \to V$ TO RESPECT L.C.'S AS WELL, I.E.,

$$\forall \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W \text{ AND SCALARS } \beta_1, \beta_2, \ldots, \beta_n,$$

$$\phi^{-1}(\beta_1\vec{w}_1 + \beta_2\vec{w}_2 + \cdots + \beta_n\vec{w}_n) = \beta_1\phi^{-1}(\vec{w}_1) + \beta_2\phi^{-1}(\vec{w}_2) + \cdots + \beta_n\phi^{-1}(\vec{w}_n)$$

[SEE PROBLEM 4(a)]

(b) THE SIGNIFICANCE OF "RESPECTING LINEAR COMBINATIONS" IS BEST SEEN VIA COMMUTATIVE DIAGRAM, AS FOLLOWS:

(IN V) (IN W)

$$\begin{tikzcd}
\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \arrow[r, "\phi"] \arrow[d, "\text{L.C. } a_1, a_2, \ldots, a_n"] & \phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_n) \arrow[d, "\text{L.C. } a_1, a_2, \ldots, a_n"] \\
a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \arrow[r, "\phi"] & a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n) = \phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n)
\end{tikzcd}$$

THE FUNCTION ϕ "COMMUTES" WITH THE OPERATION OF LINEAR COMBINATION: FORMING THE L.C. AND THEN MAPPING IT TO W VIA ϕ GIVES THE SAME RESULT AS MAPPING TO W VIA ϕ AND THEN FORMING THE L.C.!

(c) IF $\phi: V \to W$ IS AN ISOMORPHISM, THEN

$$\phi(\underbrace{\vec{0}_V}_{\text{EMPTY L.C. IN } V}) = \underbrace{\vec{0}_W}_{\text{EMPTY L.C. IN } W}.$$

2. TWO VECTOR SPACES V, W ARE <u>ISOMORPHIC</u> IF THERE IS AN ISOMORPHISM BETWEEN V AND W; IN THIS CASE, WE WRITE $V \cong W$. INFORMALLY, THIS MEANS THAT V AND W ARE MATHEMATICALLY <u>EQUIVALENT</u> AS VECTOR SPACES.

3. SUPPOSE THAT $B = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ IS A BASIS FOR V; FROM PROBLEM SET 6, WE THEN KNOW THAT $[B]: \mathbb{R}^n \to V$ IS A BIJECTIVE FUNCTION — BUT IS IT AN <u>ISOMORPHISM</u>? (I.E., DOES IT RESPECT L.C.'S?)

<u>YES</u> — SUPPOSE THAT $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m \in \mathbb{R}^n$ AND $a_1, a_2, \ldots, a_m \in \mathbb{R}$.

THEN WE CAN WRITE EACH $\vec{x}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{bmatrix}$ FOR SOME $x_{i1}, \ldots, x_{in} \in \mathbb{R}$.

NOW, JUST CHECK DIRECTLY THAT $[B]$ RESPECTS THIS L.C.:

$$[B](a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_m\vec{x}_m) = [B]\begin{bmatrix} a_1x_{11} + a_2x_{21} + \cdots + a_mx_{m1} \\ \vdots \\ a_1x_{1n} + a_2x_{2n} + \cdots + a_mx_{mn} \end{bmatrix}$$

$$= (a_1x_{11} + a_2x_{21} + \cdots + a_mx_{m1})\vec{v}_1 + (a_1x_{12} + a_2x_{22} + \cdots + a_mx_{m2})\vec{v}_2 + \cdots + (a_1x_{1n} + a_2x_{2n} + \cdots + a_mx_{mn})\vec{v}_n$$

$$= a_1(x_{11}\vec{v}_1 + x_{12}\vec{v}_2 + \cdots + x_{1n}\vec{v}_n) + a_2(x_{21}\vec{v}_1 + x_{22}\vec{v}_2 + \cdots + x_{2n}\vec{v}_n) + \cdots + a_m(x_{m1}\vec{v}_1 + x_{m2}\vec{v}_2 + \cdots + x_{mn}\vec{v}_n)$$

$$= a_1[B]\vec{x}_1 + a_2[B]\vec{x}_2 + \cdots + a_m[B]\vec{x}_m \quad \checkmark$$

4. (a) CLAIM: IF $\phi: V \to W$ IS AN ISOMORPHISM, THEN $\phi^{-1}: W \to V$ IS AN ISOMORPHISM.

PROOF: SUPPOSE THAT $\phi: V \to W$ IS AN ISOMORPHISM; THEN ϕ IS BIJECTIVE AND $\forall \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ AND SCALARS $a_1, a_2, \ldots, a_n$,

$$\phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n). \quad (!)$$

NEED TO SHOW: $\phi^{-1}: V \to W$ IS AN ISOMORPHISM, I.E., ϕ^{-1} ~~IS BIJECTIVE~~ AND $\forall \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$ AND SCALARS $a_1, a_2, \ldots, a_n$,

$$\phi^{-1}(a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n) = a_1\phi^{-1}(\vec{w}_1) + a_2\phi^{-1}(\vec{w}_2) + \cdots + a_n\phi^{-1}(\vec{w}_n)$$

LET $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$ AND SCALARS $a_1, a_2, \ldots, a_n$ BE GIVEN.

* TO USE OUR HYPOTHESIS, WE'LL NEED VECTORS $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. THE ONLY REASONABLE CHOICE IS TO TAKE $\vec{v}_1 = \phi^{-1}(\vec{w}_1)$, $\vec{v}_2 = \phi^{-1}(\vec{w}_2), \ldots, \vec{v}_n = \phi^{-1}(\vec{w}_n)$

TAKE $\vec{v}_1 = \phi^{-1}(\vec{w}_1)$, $\vec{v}_2 = \phi^{-1}(\vec{w}_2), \ldots, \vec{v}_n = \phi^{-1}(\vec{w}_n) \in V$.

THEN BY HYPOTHESIS,

$$\phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n)$$

$$= a_1\cancel{\phi(\phi^{-1}}(\vec{w}_1)) + \cdots + a_n\cancel{\phi(\phi^{-1}}(\vec{w}_n))$$

COMPARE THIS TO WHAT WE NEED — WE WANT ϕ^{-1} OF THIS, SO...

$$= a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n$$

APPLYING ϕ^{-1} TO BOTH SIDES:

$$\cancel{\phi^{-1}(\phi}(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n)) = \phi^{-1}(a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n)$$

AND USING OUR DEFINITION OF $\vec{v}_i$ ON THE LEFT-HAND SIDE:

$$a_1\phi^{-1}(\vec{w}_1) + a_2\phi^{-1}(\vec{w}_2) + \cdots + a_n\phi^{-1}(\vec{w}_n) = \phi^{-1}(a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n) \quad \blacksquare$$

(b) CLAIM: IF $\phi: V \to W$ AND $\psi: W \to U$ ARE ISOMORPHISMS, THEN $\psi \circ \phi: V \to U$ IS AN ISOMORPHISM.

PROOF: SUPPOSE THAT $\phi: V \to W$ IS AN ISOMORPHISM; THEN ϕ IS BIJECTIVE AND $\forall \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ AND SCALARS $a_1, a_2, \ldots, a_n$,

$$\phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n) \quad ①$$

ALSO SUPPOSE THAT $\psi: W \to U$ IS AN ISOMORPHISM, I.E., ψ IS BIJECTIVE AND $\forall \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$ AND SCALARS $a_1, a_2, \ldots, a_n$,

$$\psi(a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_n\vec{w}_n) = a_1\psi(\vec{w}_1) + a_2\psi(\vec{w}_2) + \cdots + a_n\psi(\vec{w}_n). \quad ②$$

NEED TO SHOW THAT $\psi \circ \phi$ IS AN ISOMORPHISM, I.E., $\psi \circ \phi$ ~~IS BIJECTIVE~~ AND $\forall \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ AND SCALARS $a_1, a_2, \ldots, a_n$,

$$(\psi \circ \phi)(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) = a_1(\psi \circ \phi)(\vec{v}_1) + a_2(\psi \circ \phi)(\vec{v}_2) + \cdots + a_n(\psi \circ \phi)(\vec{v}_n)$$

LET $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$ AND SCALARS $a_1, a_2, \ldots, a_n$ BE GIVEN. THEN:

$$(\psi \circ \phi)(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) \overset{\text{DEF}}{=} \psi(\phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n))$$

WHICH, BY HYPOTHESIS ①, $= \psi(a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_n\phi(\vec{v}_n))$

NOW TAKING $\vec{w}_1 = \phi(\vec{v}_1)$, $\vec{w}_2 = \phi(\vec{v}_2), \ldots, \vec{w}_n = \phi(\vec{v}_n)$ IN ②

THIS EQUALS $a_1\psi(\phi(\vec{v}_1)) + a_2\psi(\phi(\vec{v}_2)) + \cdots + a_n\psi(\phi(\vec{v}_n))$

$$\overset{\text{DEF}}{=} a_1(\psi \circ \phi)(\vec{v}_1) + a_2(\psi \circ \phi)(\vec{v}_2) + \cdots + a_n(\psi \circ \phi)(\vec{v}_n) \quad \blacksquare$$

(c) IF V IS AN n-DIMENSIONAL V.S., TAKE A BASIS $\mathcal{B}$ FOR V; THEN $[\mathcal{B}]: \mathbb{R}^n \to V$ IS AN ISOMORPHISM.

SIMILARLY, IF W IS ALSO AN n-DIMENSIONAL V.S., TAKE A BASIS $\mathcal{B}'$ FOR W; THEN $[\mathcal{B}']: \mathbb{R}^n \to W$ IS ALSO AN ISOMORPHISM.

[SO FAR: $V \xleftarrow{[\mathcal{B}]} \mathbb{R}^n \xrightarrow{[\mathcal{B}']} W$; INVERT $[\mathcal{B}]$ AND COMPOSE!]

NOW $[\mathcal{B}]^{-1}: V \to \mathbb{R}^n$ IS AN ISOMORPHISM BY PART (a),

SO $[\mathcal{B}'] \circ [\mathcal{B}]^{-1}: V \to W$ IS AN ISOMORPHISM BY PART (b).

$\therefore$ V AND W ARE ISOMORPHIC, BY DEFINITION. $\blacksquare$

5. [V, W: VECTOR SPACES, AND $\phi: V \to W$ AN ISOMORPHISM]

(a) CLAIM: IF $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ SPANS V, THEN $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ SPANS W.

PROOF: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ SPANS V, I.E., $\forall \vec{v} \in V$ $\exists$ SCALARS $a_1, a_2, \ldots, a_k$ WITH $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$

[NEED TO SHOW: $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ SPANS W, I.E., $\forall \vec{w} \in W$ $\exists$ SCALARS $a_1, a_2, \ldots, a_k$ WITH $\vec{w} = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k)$

WHY a'S AGAIN AND NOT β'S OR SOMETHING? BECAUSE THIS TURNS OUT TO BE A VERY SIMPLE PROOF AND THE SAME a'S WILL ACTUALLY WORK FOR BOTH.

LET $\vec{w} \in W$ BE GIVEN [OUR HYPOTHESIS WANTS A $\vec{v} \in V$ — TO GET THIS, HIT $\vec{w}$ WITH ϕ^{-1}]

THEN TAKING $\vec{v} = \phi^{-1}(\vec{w}) \in V$, OUR HYPOTHESIS GIVES US SCALARS $a_1, a_2, \ldots, a_k$ WITH $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$

[GET THIS BACK TO $\vec{w}$ BY APPLYING ϕ]

APPLYING ϕ, $\phi(\vec{v}) = \phi(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k)$
$= a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k)$

BECAUSE ϕ IS AN ISOMORPHISM

BUT $\phi(\vec{v}) = \phi(\phi^{-1}(\vec{w})) = \vec{w}$, SO WE'VE FINISHED:

$$\vec{w} = a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k)$$

THUS BY DEFINITION, $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ SPANS W. ∎

[V, W: VECTOR SPACES, AND $\phi: V \to W$ AN ISOMORPHISM]

(b) CLAIM: IF $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ IS L.I. IN V, THEN $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ IS L.I. IN W.

PROOF: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ IS L.I. IN V, I.E., $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V \Rightarrow a_1, a_2, \ldots, a_k = 0$

[NEED TO SHOW: $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ IS L.I. IN W, I.E., $a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k) = \vec{0}_W \Rightarrow a_1, a_2, \ldots, a_k = 0$

SUPPOSE $a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k) = \vec{0}_W$.

[NEED TO GET THIS BACK INTO V TO USE OUR HYPOTHESIS — APPLY ϕ^{-1}]

APPLYING ϕ^{-1}, WE THEN HAVE

$$\phi^{-1}(\underbrace{a_1\phi(\vec{v}_1) + a_2\phi(\vec{v}_2) + \cdots + a_k\phi(\vec{v}_k)}_{\text{LINEAR COMBINATION!!!}}) = \phi^{-1}(\vec{0}_W)$$

BECAUSE ϕ^{-1} IS AN ISOMORPHISM, THIS MEANS

$$a_1\cancel{\phi^{-1}(\phi}(\vec{v}_1)) + a_2\cancel{\phi^{-1}(\phi}(\vec{v}_2)) + \cdots + a_k\cancel{\phi(\phi^{-1}}(\vec{v}_k)) = \vec{0}_V$$

THUS $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V$,
SO BY HYPOTHESIS, $a_1, a_2, \ldots, a_k = 0$.

THUS BY DEFINITION, $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ IS L.I. IN W. ∎

(c) CLAIM: IF $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ IS A BASIS FOR V,
THEN $\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ IS A BASIS FOR W.

PROOF: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ IS A BASIS FOR V;
THEN BY DEFINITION, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ IS L.I. AND SPANS V.
BY PARTS (a) AND (b), THEN,
$\{\phi(\vec{v}_1), \phi(\vec{v}_2), \ldots, \phi(\vec{v}_k)\}$ IS L.I. AND SPANS W,
SO BY DEFINITION, IT IS A BASIS FOR W. ∎

* THIS TELLS US THAT ISOMORPHIC VECTOR SPACES MUST HAVE THE SAME DIMENSION!

WHY? SUPPOSE $\phi: V \to W$ IS AN ISOMORPHISM, AND THAT $\{\vec{v}_1, \ldots, \vec{v}_k\}$ IS A BASIS FOR V; PART (c) SHOWS US THAT $\{\phi(\vec{v}_1), \ldots, \phi(\vec{v}_k)\}$ IS A BASIS FOR W, SO SINCE THE DIMENSION OF A V.S. IS JUST THE SIZE OF ANY BASIS FOR IT, THESE TWO VECTOR SPACES BOTH HAVE DIMENSION k.

6. WE ALREADY KNOW THAT DIMENSION IS AN INTRINSIC PROPERTY OF A VECTOR SPACE. PROBLEM 4 TELLS US MORE, NAMELY THAT TWO VECTOR SPACES OF THE SAME DIMENSION ARE ISOMORPHIC; PROBLEM 5 TELLS US, CONVERSELY, THAT ISOMORPHIC VECTOR SPACES HAVE THE SAME DIMENSION. THUS, THE QUESTION OF VECTOR SPACE ISOMORPHISM IS SIMPLY ONE OF DIMENSION, SO ANY PROPERTY OF VECTOR SPACES THAT IS THE SAME FOR ISOMORPHIC VECTOR SPACES DEPENDS ONLY UPON DIMENSION, MAKING DIMENSION "THE" FUNDAMENTAL PROPERTY INTRINSIC TO A VECTOR SPACE, IF WE CONSIDER ISOMORPHIC VECTOR SPACES TO BE EQUIVALENT.

7. $\left[\mathcal{B} = (1-x,\ x+2x^2,\ 3+x^2) : \text{BASIS FOR } P_2(x);\ \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \in \mathbb{R}^3\right]$

$$[\mathcal{B}](2\vec{x}_1 - 3\vec{x}_2) = [\mathcal{B}]\left(\begin{bmatrix} -2 \\ 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix}\right) = [\mathcal{B}]\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$
$$= -5(1-x) + 0(x+2x^2) + 1(3+x^2) = \underline{-2 + 5x + x^2}.$$

$$2[\mathcal{B}]\vec{x}_1 - 3[\mathcal{B}]\vec{x}_2 = 2[\mathcal{B}]\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} - 3[\mathcal{B}]\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$= 2\left[-1(1-x) + 3(x+2x^2) + 2(3+x^2)\right] - 3\left[1(1-x) + 2(x+2x^2) + 1(3+x^2)\right]$$
$$= -2(1-x) + 6(x+2x^2) + 4(3+x^2) - 3(1-x) - 6(x+2x^2) - 3(3+x^2)$$
$$= \underline{-2 + 5x + x^2} \quad \checkmark$$

8. $\left[\mathcal{B} = (1, x, x^2) : \text{BASIS FOR } P_2(x);\ \vec{v}_1 = 3x^2 + x + 2,\ \vec{v}_2 = x - x^2 + 1 \in P_2(x)\right]$

$$[\mathcal{B}]^{-1}(-4\vec{v}_1 + 2\vec{v}_2) = [\mathcal{B}]^{-1}\left(-4(3x^2+x+2) + 2(x - x^2 + 1)\right)$$
$$= [\mathcal{B}]^{-1}(-14x^2 - 2x - 6)$$
$$= [\mathcal{B}]^{-1}(-6 \cdot 1 - 2x - 14x^2) = \underline{\begin{bmatrix} -6 \\ -2 \\ -14 \end{bmatrix}}$$

$$-4[\mathcal{B}]^{-1}\vec{v}_1 + 2[\mathcal{B}]^{-1}\vec{v}_2 = -4[\mathcal{B}]^{-1}(3x^2 + x + 2) + 2[\mathcal{B}]^{-1}(x - x^2 + 1)$$
$$= -4[\mathcal{B}]^{-1}(2 \cdot 1 + 1x + 3x^2) + 2[\mathcal{B}]^{-1}(1 \cdot 1 + 1 \cdot x - 1 \cdot x^2)$$
$$= -4\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \underline{\begin{bmatrix} -6 \\ -2 \\ -14 \end{bmatrix}} \quad \checkmark$$

9. $\mathcal{B}$: BASIS FOR 4-DIMENSIONAL V.S. V; $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5 \in V$ HAVE

$$[\mathcal{B}]^{-1}\vec{v}_1 = \begin{bmatrix}1\\1\\4\\6\end{bmatrix}; \quad [\mathcal{B}]^{-1}\vec{v}_2 = \begin{bmatrix}0\\1\\5\\2\end{bmatrix}; \quad [\mathcal{B}]^{-1}\vec{v}_3 = \begin{bmatrix}2\\3\\9\\18\end{bmatrix};$$

$$[\mathcal{B}]^{-1}\vec{v}_4 = \begin{bmatrix}-2\\0\\2\\-8\end{bmatrix}; \quad [\mathcal{B}]^{-1}\vec{v}_5 = \begin{bmatrix}4\\2\\6\\14\end{bmatrix}.$$

* USE $[\mathcal{B}], [\mathcal{B}]^{-1}$ TO TRANSLATE PROBLEMS!

(a) BY ISOMORPHISM (SEE PROBLEM 5(b)), THESE VECTORS WILL BE L.I. IN V IF, AND ONLY IF, THEIR COORDINATES ARE L.I. IN $\mathbb{R}^4$:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2 & -2 & 4 \\ 1 & 1 & 3 & 0 & 2 \\ 4 & 5 & 9 & 2 & 6 \\ 6 & 2 & 18 & -8 & 14 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{array}{ccccc} & & & \text{FREE} & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \to \text{NOT L.I.! } (\exists \text{ A NON-PIVOT COLUMN})$$

WE CAN EASILY FIND A LINEAR RELATION ON THE COORDINATE VECTORS BY SOLVING THE SYSTEM:

$$\begin{cases} a_4 : \text{FREE} \\ a_1 = 2a_4 \\ a_2 = -2a_4 \\ a_3 = 0 \\ a_5 = 0 \end{cases}$$

TO MAKE THIS NONTRIVIAL, TAKE $a_4 = 1$ (E.G.):

$a_4 = 1, a_1 = 2, a_2 = -2, a_3 = a_5 = 0$

$$\text{SO } 2\begin{bmatrix}1\\1\\4\\6\end{bmatrix} - 2\begin{bmatrix}0\\1\\5\\2\end{bmatrix} + 0\begin{bmatrix}2\\3\\9\\18\end{bmatrix} + 1\begin{bmatrix}-2\\0\\2\\-8\end{bmatrix} + 0\begin{bmatrix}4\\2\\6\\14\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

APPLYING THE ISOMORPHISM $[\mathcal{B}]$, WE KNOW THAT

$$\underbrace{2\vec{v}_1 - 2\vec{v}_2 + 0\vec{v}_3 + \vec{v}_4 + 0\vec{v}_5 = \vec{0}_V}.$$

(b) SEE THE MATRIX REDUCTION ABOVE — BY ISOMORPHISM (SEE PROBLEM 5(a)), THESE VECTORS DO SPAN V, BECAUSE THEIR COORDINATE VECTORS SPAN $\mathbb{R}^4$ (PIVOTS IN ALL ROWS!).

(c) AGAIN, BY ISOMORPHISM (SEE PROBLEM 5(c)), SINCE WE CAN REMOVE THE FOURTH COORDINATE VECTOR TO GET A BASIS FOR $\mathbb{R}^4$, THE COLLECTION $\underbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}}$ IS A BASIS FOR V.

* WE CAN DO THIS ALL INDEPENDENT OF WHAT THE VECTORS OF V ACTUALLY REPRESENT (HERE, WE DON'T EVEN KNOW!), BECAUSE WE CAN TRANSLATE OUR QUESTIONS VIA ISOMORPHISM TO SIMPLE QUESTIONS ABOUT COLUMN VECTORS! ☺

(THIS IS WHY THE FTVS IS, IN PRACTICE, SO FUNDAMENTAL!)