1. ($\mathcal{B}$: ORDERED BASIS FOR AN n-DIMENSIONAL VECTOR SPACE V)

(a) IF $\vec{x} \in \mathbb{R}^n$, THE VECTOR $[\mathcal{B}]\vec{x} \in V$ IS THE LINEAR COMBINATION OF $\mathcal{B}$ IN WHICH THE COEFFICIENT OF THE i^{TH} VECTOR IS GIVEN BY THE i^{TH} ENTRY OF $\vec{x}$.

I.E., IF $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ AND $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

THEN $[\mathcal{B}]\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$.

(b) THE FACT THAT $\mathcal{B}$ IS A BASIS TELLS US THAT EVERY VECTOR OF V CAN BE WRITTEN UNIQUELY AS A LINEAR COMBINATION OF $\mathcal{B}$ — THUS, BY THE DEFINITION OF $[\mathcal{B}]$, EACH VECTOR $\vec{v} \in V$ HAS EXACTLY ONE VECTOR $x \in \mathbb{R}^n$ WITH $[\mathcal{B}]\vec{x} = \vec{v}$; CONVERSELY, EACH $\vec{x} \in \mathbb{R}^n$ DETERMINES A UNIQUE VECTOR $[\mathcal{B}]\vec{x} \in V$, SO $[\mathcal{B}]$ PAIRS EACH $\vec{x} \in \mathbb{R}^n$ WITH SOME $\vec{v} \in V$ AND VICE-VERSA.

(c) IF $\vec{v} \in V$, THE COLUMN VECTOR $[\mathcal{B}]^{-1}\vec{v}$ TELLS US WHAT COEFFICIENTS TO TAKE TO FORM $\vec{v}$ AS A LINEAR COMBINATION OF $\mathcal{B}$; THESE ARE CALLED THE COORDINATES FOR $\vec{v}$ WITH RESPECT TO $\mathcal{B}$, OR SIMPLY THE $\mathcal{B}$-COORDINATES OF $\vec{v}$.

[WE CALL $[\mathcal{B}]^{-1}\vec{v}$ A COORDINATE VECTOR.]

* EACH ENTRY OF $[\mathcal{B}]^{-1}\vec{v}$ GIVES US THE COEFFICIENT OF THE CORRESPONDING VECTOR OF $\mathcal{B}$ WHEN WE WRITE $\vec{v}$ AS A L.C. OF $\mathcal{B}$.

2. WHEN DEALING WITH COORDINATES AND COORDINATE VECTORS, WE SHOULD ALWAYS KEEP IN MIND THAT THE EACH NUMBER CORRESPONDS TO A PARTICULAR VECTOR OF A PARTICULAR BASIS — THEY ARE NOT JUST FREE-FLOATING VALUES; EACH COORDINATE VECTOR IS ATTACHED TO SOME BASIS $\mathcal{B}$.

3. ($\mathcal{B}$: ORDERED BASIS FOR $\mathbb{R}^n$)

(a) FOR A GIVEN COORDINATE VECTOR $\vec{x} \in \mathbb{R}^n$, $[\mathcal{B}]\vec{x}$ IS SIMPLY THE LINEAR COMBINATION OF $\mathcal{B}$ SPECIFIED BY $\vec{x}$ (WITH THE ENTRIES OF $\vec{x}$ GIVING THE COEFFICIENTS OF THE VECTORS IN $\mathcal{B}$).

E.G., IF $\mathcal{B} = \left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$ FOR $\mathbb{R}^3$ AND $\vec{x} = \begin{bmatrix}3\\0\\2\end{bmatrix}$,

THEN $[\mathcal{B}]\vec{x} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}4\\0\\1\end{bmatrix} + 2\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\8\\9\end{bmatrix}$.

(b) GIVEN SOME VECTOR $\vec{v} \in \mathbb{R}^n$, ITS $\mathcal{B}$-COORDINATES $[\mathcal{B}]^{-1}\vec{v}$ CAN BE COMPUTED BY SOLVING THE LINEAR SYSTEM $[\mathcal{B} \mid \vec{v}]$. (AFTER ALL, WE'RE TRYING TO WRITE $\vec{v}$ AS A L.C. OF $\mathcal{B}$!)

E.G., TO FIND $[\mathcal{B}]^{-1}\begin{bmatrix}1\\8\\9\end{bmatrix}$ ($\mathcal{B}$ AS ABOVE),

WE SOLVE $\left[\mathcal{B} \,\middle|\, \begin{matrix}1\\8\\9\end{matrix}\right]$ TO OBTAIN $[\mathcal{B}]^{-1}\begin{bmatrix}1\\8\\9\end{bmatrix} = \begin{bmatrix}3\\0\\2\end{bmatrix}$.

4. THE STANDARD BASIS FOR $\mathbb{R}^n$ IS $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$,

WHERE $\vec{e}_i = \begin{bmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{bmatrix}$ ← i^{TH} ENTRY IS 1, WITH ALL OTHER ENTRIES 0.

A QUICK LOOK SHOWS US THAT THE STANDARD COORDINATES FOR ANY $\vec{v} \in \mathbb{R}^n$ ARE JUST THE VECTOR $\vec{v}$!

E.G., VECTOR $\begin{bmatrix}1\\2\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix} \longleftrightarrow$ COORDINATES $\begin{bmatrix}1\\2\\3\end{bmatrix}$

5. $P_n(x)$ IS THE SUBSPACE OF $\mathbb{R}[x]$ CONSISTING OF ALL POLYNOMIALS HAVING DEGREE $\leq n$.

(IN SYMBOLS, $P_n(x) = \{A_0 + A_1x + \cdots + A_nx^n : A_0, A_1, \ldots, A_n \in \mathbb{R}\}$.)

THE EASIEST BASIS FOR $P_n(x)$ IS $(1, x, x^2, \ldots, x^n)$,

SO $\dim P_n(x) = n+1$.

6. IF $\mathcal{B}=\left(\begin{bmatrix}0\\1\\3\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\1\\1\end{bmatrix}\right)$:

(a) $[\mathcal{B}]\begin{bmatrix}4\\1\\3\end{bmatrix}=4\begin{bmatrix}0\\1\\3\end{bmatrix}+1\begin{bmatrix}1\\2\\-1\end{bmatrix}+3\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}7\\9\\14\end{bmatrix}$

(b) $[\mathcal{B}]\begin{bmatrix}-1\\0\\2\end{bmatrix}=-1\begin{bmatrix}0\\1\\3\end{bmatrix}+0\begin{bmatrix}1\\2\\-1\end{bmatrix}+2\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}4\\1\\-1\end{bmatrix}$

(c) $[\mathcal{B}]\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}0\\1\\3\end{bmatrix}+0\begin{bmatrix}1\\2\\-1\end{bmatrix}+0\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) $[\mathcal{B}]\begin{bmatrix}0\\1\\0\end{bmatrix}=0\begin{bmatrix}0\\1\\3\end{bmatrix}+1\begin{bmatrix}1\\2\\-1\end{bmatrix}+0\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$

(e) $\left[\begin{array}{ccc|c}0&1&2&4\\1&2&1&1\\3&-1&1&3\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-2/3\\0&0&1&7/3\end{array}\right]$ $\therefore [\mathcal{B}]^{-1}\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}0\\-2/3\\7/3\end{bmatrix}$

(f) $\left[\begin{array}{ccc|c}0&1&2&-1\\1&2&1&0\\3&-1&1&2\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&3/4\\0&1&0&-1/6\\0&0&1&-5/12\end{array}\right]$ $\therefore [\mathcal{B}]^{-1}\begin{bmatrix}4\\1\\3\end{bmatrix}=\begin{bmatrix}3/4\\-1/6\\-5/12\end{bmatrix}$

(g) $\left[\begin{array}{ccc|c}0&1&2&7\\1&2&1&9\\3&-1&1&14\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&4\\0&1&0&1\\0&0&1&3\end{array}\right]$ $\therefore [\mathcal{B}]^{-1}\begin{bmatrix}7\\9\\14\end{bmatrix}=\begin{bmatrix}4\\1\\3\end{bmatrix}$

(COMPARE WITH PART (a)!)

(h) $\left[\begin{array}{ccc|c}0&1&2&4\\1&2&1&1\\3&-1&1&-1\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&0\\0&0&1&2\end{array}\right]$ $\therefore [\mathcal{B}]^{-1}\begin{bmatrix}4\\1\\-1\end{bmatrix}=\begin{bmatrix}-1\\0\\2\end{bmatrix}$

(COMPARE WITH PART (b)!)

7. IF $\mathcal{B}$ IS AN ORDERED BASIS FOR AN n-DIMENSIONAL VECTOR SPACE V:

- $\forall\ \vec{v}\in V,\quad [\mathcal{B}][\mathcal{B}]^{-1}\vec{v}=\vec{v}$

AND
- $\forall\ x\in\mathbb{R}^n,\quad [\mathcal{B}]^{-1}[\mathcal{B}]\vec{x}=\vec{x}$

$[\mathcal{B}]$ AND $[\mathcal{B}]^{-1}$ ARE INVERSE FUNCTIONS

THE OPERATIONS OF $[\mathcal{B}]$ ("TAKE THIS L.C. OF $\mathcal{B}$") AND $[\mathcal{B}]^{-1}$ ("WHAT L.C. OF $\mathcal{B}$ GIVES ME THIS?") UNDO EACH OTHER!

8. (RECALL THAT WE FIND $\mathcal{B}$-COORDINATES FOR A COLUMN VECTOR $\vec{v}$ BY SOLVING $[\mathcal{B}\,|\,\vec{v}]$)

(a) $\mathcal{B}'=(\vec{e}_2,\vec{e}_1,\vec{e}_3)=\left(\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$;

$\left[\begin{array}{ccc|c}0&1&0&-2\\1&0&0&4\\0&0&1&-3\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&4\\0&1&0&-2\\0&0&1&-3\end{array}\right]$ $\therefore [\mathcal{B}]^{-1}\begin{bmatrix}-2\\4\\-3\end{bmatrix}=\begin{bmatrix}4\\-2\\-3\end{bmatrix}$

(b) $\mathcal{B}''=(6\vec{e}_1,\vec{e}_2,\vec{e}_3)=\left(\begin{bmatrix}6\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$;

$\cdot\frac{1}{6}\rightarrow\left[\begin{array}{ccc|c}6&0&0&-2\\0&1&0&4\\0&0&1&-3\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&-1/3\\0&1&0&4\\0&0&1&-3\end{array}\right]$ $\therefore [\mathcal{B}'']^{-1}\begin{bmatrix}-2\\4\\-3\end{bmatrix}=\begin{bmatrix}-1/3\\4\\-3\end{bmatrix}$

(c) $\mathcal{B}'''=(\vec{e}_1+4\vec{e}_2,\vec{e}_2,\vec{e}_3)=\left(\begin{bmatrix}1\\4\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$;

$-4\left[\begin{array}{ccc|c}1&0&0&-2\\4&1&0&4\\0&0&1&-3\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&-2\\0&1&0&12\\0&0&1&-3\end{array}\right]$ $\therefore [\mathcal{B}''']^{-1}\begin{bmatrix}-2\\4\\-3\end{bmatrix}=\begin{bmatrix}-2\\12\\-3\end{bmatrix}$

9. $\mathcal{B}=(\vec{a},\vec{b},\vec{c})$: BASIS FOR A VECTOR SPACE V; $\vec{v}\in V$ WITH $[\mathcal{B}]^{-1}\vec{v}=\begin{bmatrix}-1\\3\\2\end{bmatrix}$

* NOTE THAT THIS MEANS $\vec{v}=[\mathcal{B}]\begin{bmatrix}-1\\3\\2\end{bmatrix}=-\vec{a}+3\vec{b}+2\vec{c}$

(a) WE NEED TO SOLVE $x_1\vec{b}+x_2\vec{c}+x_3\vec{a}=\vec{v}=-\vec{a}+3\vec{b}+2\vec{c}$

I.E., $(x_1-3)\vec{b}+(x_2-2)\vec{c}+(x_3+1)\vec{a}=\vec{0}$

BECAUSE $(\vec{a},\vec{b},\vec{c})$ IS A BASIS, IT'S L.I.,

SO THIS IMPLIES THAT $x_1-3=0$, $x_2-2=0$, $x_3+1=0$,

I.E., $x_1=3$, $x_2=2$, $x_3=-1$.

$\therefore$ THE COORDINATE VECTOR IS $\begin{bmatrix}3\\2\\-1\end{bmatrix}$

(b) SOLVE $x_1(2\vec{a})+x_2(-\vec{b})+x_3(\vec{c})=\vec{v}=-\vec{a}+3\vec{b}+2\vec{c}$

I.E., $(2x_1+1)\vec{a}+(-x_2-3)\vec{b}+(x_3-2)\vec{c}=0$

AGAIN BY LINEAR INDEPENDENCE, WE THEN HAVE

$2x_1+1=0$, $-x_2-3=0$, $x_3-2=0$

SO $x_1=-\frac{1}{2}$, $x_2=-3$, $x_3=2$

$\therefore$ THE COORDINATE VECTOR IS $\begin{bmatrix}-1/2\\-3\\2\end{bmatrix}$

(c) SOLVE $x_1(\vec{a}+2\vec{b}-3\vec{c})+x_2\vec{b}+x_3\vec{c}=\vec{v}=-\vec{a}+3\vec{b}+2\vec{c}$

I.E., $x_1\vec{a}+2x_1\vec{b}-3x_1\vec{c}+x_2\vec{b}+x_3\vec{c}+\vec{a}-3\vec{b}-2\vec{c}=0$

OR, $(x_1+1)\vec{a}+(2x_1+x_2-3)\vec{b}+(-3x_1+x_3-2)\vec{c}=0$

BY LINEAR INDEPENDENCE, $x_1+1=0$, SO $x_1=-1$

$2x_1+x_2-3=0$, SO $x_2=3-2x_1=5$

AND $-3x_1+x_3-2=0$, SO $x_3=2+3x_1=-1$

$\therefore$ THE COORDINATE VECTOR IS $\begin{bmatrix}-1\\5\\-1\end{bmatrix}$

10. $\mathcal{B}=(1-t^2,\ t+t^2,\ 2t)$; $\mathcal{B}'=(2-t+t^2,\ 2+t^2,\ 1+t+t^2)$

(a) IF $f(t)$ HAS $\mathcal{B}$-COORDINATES $\begin{bmatrix}1\\2\\3\end{bmatrix}$, THEN BY DEFINITION OF COORDINATES, $f(t)=[\mathcal{B}]\begin{bmatrix}1\\2\\3\end{bmatrix}=1(1-t^2)+2(t+t^2)+3(2t)$

$=1-t^2+2t+2t^2+6t$

$=1+8t+t^2$

(b) TO FIND THE $\mathcal{B}'$-COORDINATES FOR $f(t)$, SOLVE:

$x_1(2-t+t^2)+x_2(2+t^2)+x_3(1+t+t^2)=1+8t+t^2$

I.E., $2x_1-x_1t+x_1t^2+2x_2+x_2t^2+x_3+x_3t+x_3t^2=1+8t+t^2$

I.E., $(2x_1+2x_2+x_3)+(-x_1+x_3)t+(x_1+x_2+x_3)t^2=1+8t+t^2$

EQUATING COEFFICIENTS OF 1, t, AND t^2:
$$\begin{aligned}2x_1+2x_2+x_3&=1\\-x_1\qquad\ +x_3&=8\\x_1+x_2+x_3&=1\end{aligned}$$

$$\begin{array}{ccc|c}x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c}2&2&1&1\\-1&0&1&8\\1&1&1&1\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&0&0&-7\\0&1&0&7\\0&0&1&1\end{array}\right],$$

SO THE $\mathcal{B}'$-COORDINATES OF $f(t)$ ARE $\begin{bmatrix}-7\\7\\1\end{bmatrix}$