1. ($\mathcal{B}$: COLLECTION OF VECTORS IN A FINITE-DIMENSIONAL VECTOR SPACE V.)

$\mathcal{B}$ IS A BASIS FOR V IF ① $\mathcal{B}$ IS LINEARLY INDEPENDENT IN V AND ② $\mathcal{B}$ SPANS V.

BECAUSE $\mathcal{B}$ SPANS V

* THIS TELLS US THAT EVERY VECTOR $\vec{v} \in V$ CAN BE WRITTEN UNIQUELY AS A LINEAR COMBINATION OF $\mathcal{B}$.

BECAUSE $\mathcal{B}$ IS L.I. IN V

2. IF $\mathcal{B}$ AND $\mathcal{B}'$ ARE FINITE BASES FOR A VECTOR SPACE V, THEN $|\mathcal{B}| = |\mathcal{B}'|$, BECAUSE:

- ON THE ONE HAND, $\mathcal{B}$ SPANS V AND $\mathcal{B}'$ IS L.I. IN V, SO $|\mathcal{B}| \geq |\mathcal{B}'|$.
- ON THE OTHER HAND, $\mathcal{B}$ IS L.I. IN V AND $\mathcal{B}'$ SPANS V, SO $|\mathcal{B}| \leq |\mathcal{B}'|$.

(NET EFFECT: $|\mathcal{B}| = |\mathcal{B}'|$)

* THIS TELLS US THAT WE CAN DEFINE THE DIMENSION OF V TO BE THE SIZE OF ANY BASIS FOR V; THIS QUANTITY IS INTRINSIC TO V (I.E., DEPENDS ONLY UPON V, NOT ANY ARBITRARY CHOICES WE MADE IN DEFINING IT), BECAUSE IN LIGHT OF THE ABOVE REMARK, IT DOESN'T MATTER WHICH BASIS WE USE TO DEFINE IT (ALL BASES FOR V HAVE THE SAME SIZE!).

3. IF V IS A FINITE-DIMENSIONAL VECTOR SPACE:

(a) ANY FINITE SPANNING SET $\mathcal{C}$ FOR V CAN BE REDUCED TO A BASIS FOR V BY SUCCESSIVELY REMOVING ANY VECTOR OF $\mathcal{C}$ THAT IS A L.C. OF THE REST UNTIL NO SUCH VECTORS ARE LEFT IN THE COLLECTION.

(REMOVING A VECTOR THAT'S A L.C. OF THE REST DOESN'T CHANGE THE SPAN OF $\mathcal{C}$; WHEN THERE ARE NO SUCH VECTORS LEFT, THEN OUR SPANNING SET $\mathcal{C}$ IS LINEARLY INDEPENDENT, AS WELL, SO IT HAS BEEN MADE INTO A BASIS.)

(b) ANY LINEARLY INDEPENDENT SET $\mathcal{C}$ IN V CAN BE EXTENDED TO A BASIS FOR V BY SUCCESSIVELY ADDING A VECTOR $\vec{v} \in V$ TO $\mathcal{C}$ THAT'S NOT IN SPAN($\mathcal{C}$).

(ADDING A VECTOR NOT IN SPAN($\mathcal{C}$) TO $\mathcal{C}$ PRESERVES THE LINEAR INDEPENDENCE OF $\mathcal{C}$; WHEN NO SUCH VECTORS REMAIN, OUR L.I. SET $\mathcal{C}$ WILL HAVE SPAN($\mathcal{C}$) $= V$ — THUS IT WILL SPAN V AS WELL, SO IT HAS BEEN MADE INTO A BASIS)

(c) WE KNOW THAT V HAS A BASIS, BECAUSE WE COULD START WITH THE EMPTY COLLECTION $\mathcal{C} = \{\}$ IN V (WHICH IS TRIVIALLY LINEARLY INDEPENDENT) AND EXTEND IT TO A BASIS FOR V, AS IN PART (a).

4. Suppose that $\mathcal{V}$ is an n-dimensional vector space.
↳ $\therefore$ any basis for $\mathcal{V}$ has n vectors

(a) If $\mathcal{C}$ spans $\mathcal{V}$ and $|\mathcal{C}| = n$, then we know $\mathcal{C}$ can be reduced to a basis, which will have n vectors.
— But $\mathcal{C}$ already has just n vectors, so it must already be a basis for $\mathcal{V}$!

(b) If $\mathcal{D}$ is linearly independent in $\mathcal{V}$ and has $|\mathcal{D}| = n$, then we know $\mathcal{D}$ can be extended into a basis, which will have n vectors.
— But $\mathcal{D}$ already has n vectors, so it must already be a basis for $\mathcal{V}$

* Net effect: If we know the dimension of our vector space, a spanning set of the correct size is automatically L.I., and an L.I. collection of the correct size automatically spans! — So, for a collection of vectors of the right size, we only need to show half of what it means to be a basis, and the other half comes for free!

5. If $\mathcal{B}$ is a basis [of column vectors for] $\mathbb{R}^m$, then every linear system arising from $\mathcal{B}$ is consistent and has a unique solution.

Why? Recall that for column vectors, linear independence means that every column has a pivot ($\therefore$ no free variables) and spanning means that every row has a pivot ($\therefore$ consistency) — a basis has both properties!

6. Any linearly independent collection $\mathcal{C}$ in $\mathbb{R}^m$ can be computationally extended to a basis by appending any spanning set to $\mathcal{C}$ and reducing this to a basis as usual (form a matrix, reduce it, and take the vectors corresponding to the pivot columns).

E.G., $$\left[\begin{array}{c|cccccc} & 1 & 0 & 0 & & 0 & 0 \\ & 0 & 1 & 0 & & \vdots & \vdots \\ \mathcal{C} & 0 & 0 & 1 & \cdots & & \\ & \vdots & \vdots & \vdots & & 0 & 0 \\ & & & & & 1 & 0 \\ & 0 & 0 & 0 & & 0 & 1 \end{array}\right]$$ → Reduce + find pivots to determine a basis

(Note that because $\mathcal{C}$ is L.I., each column of $\mathcal{C}$ will give a pivot — the remaining pivots tell us which vectors to insert to get a basis for $\mathbb{R}^m$.)

7. Suppose that $\mathcal{V}$ is a 7-dimensional V.S.
↳ $\therefore$ it has a basis $\mathcal{B}$ with $|\mathcal{B}| = 7$.

(a) If $\mathcal{C}$ spans $\mathcal{V}$, then $\underset{\text{(spans)}}{|\mathcal{C}|} \geq \underset{\text{(L.I.)}}{|\mathcal{B}|} = 7$, so $|\mathcal{C}| \geq 7$.

(b) If $\mathcal{D}$ is L.I. in $\mathcal{V}$, then $\underset{\text{(L.I.)}}{|\mathcal{D}|} \leq \underset{\text{(spans)}}{|\mathcal{B}|} = 7$, so $|\mathcal{D}| \leq 7$.

8. SUPPOSE THAT ① $\mathcal{C}$ SPANS V, WITH $|\mathcal{C}|=5$
② $\mathcal{D}$ IS L.I. IN V, WITH $|\mathcal{D}|=3$

(a) TO DETERMINE THE POSSIBLE DIMENSIONS FOR V, SUPPOSE THAT $\mathcal{B}$ IS A BASIS FOR V.

THEN $\underset{\text{(L.I.)}}{3=|\mathcal{D}|} \le \underset{\text{(SPANS + L.I.)}}{|\mathcal{B}|} \le \underset{\text{(SPANS)}}{|\mathcal{C}|=5}$

THUS, THE DIMENSION OF V, WHICH EQUALS $|\mathcal{B}|$, MUST BE 3, 4, OR 5.

(b) IF $\dim V = 3$:

- $\mathcal{C}$ CAN'T BE A BASIS ($|\mathcal{C}|=5\neq 3$), SO SINCE WE ALREADY KNOW $\mathcal{C}$ SPANS V, $\mathcal{C}$ CAN'T BE L.I.
- $\mathcal{D}$ IS A LINEARLY INDEPENDENT SET OF THE RIGHT SIZE ($|\mathcal{D}|=3$), SO IT MUST, IN FACT, BE A BASIS.

IF $\dim V = 4$:

- $\mathcal{C}$ CAN'T BE A BASIS ($|\mathcal{C}|=5\neq 4$), SO SINCE WE ALREADY KNOW $\mathcal{C}$ SPANS V, $\mathcal{C}$ CAN'T BE L.I.
- $\mathcal{D}$ CAN'T BE A BASIS EITHER ($|\mathcal{D}|=3\neq 4$), SO SINCE WE ALREADY KNOW THAT $\mathcal{D}$ IS L.I., $\mathcal{D}$ CAN'T SPAN V.

IF $\dim V = 5$:

- $\mathcal{C}$ IS A SPANNING SET FOR V OF THE RIGHT SIZE ($|\mathcal{C}|=5$), SO $\mathcal{C}$ MUST BE A BASIS FOR V.
- $\mathcal{D}$ CAN'T BE A BASIS ($|\mathcal{D}|=3\neq 5$), SO SINCE WE ALREADY KNOW THAT $\mathcal{D}$ IS L.I., $\mathcal{D}$ CAN'T SPAN V.

9. V: 3-DIMENSIONAL V.S., SUCH THAT: ① $\{\vec{a},\vec{b},\vec{c},\vec{d}\}$ SPANS V,
② $\{\vec{a},\vec{c}\}$ IS L.I. IN V,
AND ③ $\{\vec{a},\vec{b},\vec{c}\}$ IS LINEARLY DEPENDENT IN V.

CLAIM: $\vec{b}\in \operatorname{SPAN}\{\vec{a},\vec{c}\}$ (NEED TO FIND α_1,α_2 WITH $\vec{b}=\alpha_1\vec{a}+\alpha_2\vec{c}$)

PROOF: (③ IS THE HYPOTHESIS INVOLVING $\vec{a},\vec{b},\vec{c}$ AND EXISTENCE OF SCALARS, SO LET'S TRY IT:)

③ $\Rightarrow \exists$ SCALARS β_1,β_2,β_3, NOT ALL ZERO, SUCH THAT

$$\beta_1\vec{a}+\beta_2\vec{b}+\beta_3\vec{c}=\vec{0}$$

(IF WE CAN SHOW THAT $\beta_2\neq 0$, WE CAN SOLVE THIS FOR $\vec{b}$!)

TAKE SUCH β_1,β_2,β_3;

THEN $\beta_2\neq 0$, BECAUSE IF IT WERE, WE'D HAVE $\beta_1\vec{a}+\beta_3\vec{c}=\vec{0}$ WITH β_1,β_3 NOT BOTH ZERO — THIS WOULD BE A NONTRIVIAL LINEAR RELATION ON $\{\vec{a},\vec{c}\}$, CONTRADICTING ②!

SINCE $\beta_2\neq 0$, WE CAN SOLVE FOR $\vec{b}$:

$$\beta_1\vec{a}+\beta_2\vec{b}+\beta_3\vec{c}=\vec{0}$$
$$\Rightarrow \beta_2\vec{b}=(-\beta_1)\vec{a}+(-\beta_3)\vec{c}$$
$$\Rightarrow \vec{b}=\left(-\frac{\beta_1}{\beta_2}\right)\vec{a}+\left(-\frac{\beta_3}{\beta_2}\right)\vec{c}.$$

TAKING $\alpha_1=-\frac{\beta_1}{\beta_2}$ AND $\alpha_2=-\frac{\beta_3}{\beta_2}$,

WE THUS HAVE $\vec{b}=\alpha_1\vec{a}+\alpha_2\vec{c}$, SO $\vec{b}\in\operatorname{SPAN}\{\vec{a},\vec{c}\}$ ■

(NOW, FINDING A BASIS FOR V ISN'T SO HARD; $\dim V=3$, SO IF WE CAN REMOVE A VECTOR FROM THE SPANNING SET $\{\vec{a},\vec{b},\vec{c},\vec{d}\}$ WITHOUT AFFECTING ITS SPAN, WE'LL HAVE A BASIS... BUT WE JUST SHOWED THAT $\vec{b}$ WAS IN THE SPAN OF THE REST, SO WE CAN REMOVE IT WHILE PRESERVING THE SPAN!)

CLAIM: $\{\vec{a},\vec{c},\vec{d}\}$ IS A BASIS FOR V

PROOF: BY HYPOTHESIS, $\{\vec{a},\vec{b},\vec{c},\vec{d}\}$ SPANS V;
SINCE $\vec{b}$ IS IN THE SPAN OF $\{\vec{a},\vec{c}\}$,
WE CAN REMOVE IT AND STILL HAVE $\{\vec{a},\vec{c},\vec{d}\}$ SPAN V.
BUT THEN $\{\vec{a},\vec{c},\vec{d}\}$ IS A SPANNING SET OF THE RIGHT SIZE, SO IT MUST BE A BASIS FOR V ■

10. IF W IS A SUBSPACE OF A FINITE-DIMENSIONAL VECTOR SPACE V, THEN $\underbrace{\dim W \le \dim V}$.

WHY? TAKE A BASIS $\mathcal{B}$ FOR W; THEN $\mathcal{B}$ IS A L.I. SET IN V, SO WE CAN EXTEND IT TO A BASIS $\mathcal{B}'$ OF V.
THEN $\mathcal{B} \subset \mathcal{B}'$, SO $|\mathcal{B}| \le |\mathcal{B}'|$, SO BY DEFINITION OF DIMENSION, $\dim W \le \dim V$ ✓

11. TO FIND THE DIMENSION OF $\{\vec{0}\}$, WE NEED TO FIND A BASIS FOR $\{\vec{0}\}$ — I.E., A L.I. COLLECTION IN $\{\vec{0}\}$ THAT SPANS $\{\vec{0}\}$. A L.I. COLLECTION CAN'T CONTAIN THE ZERO VECTOR, SO OUR ONLY HOPE IS $\mathcal{B} = \{\,\}$ (THE EMPTY COLLECTION). THIS ACTUALLY WORKS, IF WE CAREFULLY CHECK THE DEFINITIONS! (RECALL THAT A NONTRIVIAL LINEAR RELATION MUST HAVE AT LEAST ONE NONZERO COEFFICIENT, AND THAT THE L.C. OF NO VECTORS AT ALL RESULTS IN $\vec{0}$).
THUS, $\underbrace{\dim\{\vec{0}\} = |\{\,\}| = 0}$! (MOSTLY A THOUGHT EXERCISE)

12. (EXTEND TO A SPANNING SET, THEN REDUCE THAT TO A BASIS...)

$$\left\{ \begin{bmatrix} -1 \\ -5 \\ 12 \\ 11 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 38 \\ 36 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 7 \\ 7 \end{bmatrix}, \overset{①}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \overset{②}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{③}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{④}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}:$$

$$\begin{array}{c} \begin{array}{ccccccc} & & & ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} -1 & -3 & -1 & 1 & 0 & 0 & 0 \\ -5 & -16 & -3 & 0 & 1 & 0 & 0 \\ 12 & 38 & 7 & 0 & 0 & 1 & 0 \\ 11 & 36 & 7 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccccccc} & & & ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 15/7 & -13/7 \\ 0 & 1 & 0 & 1 & 0 & -4/7 & 5/7 \\ 0 & 0 & 1 & -2 & 0 & -3/7 & 2/7 \\ 0 & 0 & 0 & 0 & 1 & 2/7 & 1/7 \end{bmatrix} \end{array}$$

DOESN'T ACTUALLY MATTER — WE ONLY NEED TO FIND THE PIVOTS!

$$\therefore \underbrace{\left\{ \begin{bmatrix} -1 \\ -5 \\ 12 \\ 11 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 38 \\ 36 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$ IS A BASIS FOR $\mathbb{R}^4$.

13. (SAME GAME AS #12)

$$\begin{array}{c} \begin{array}{ccccccc} & & & ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & -4 & 6 & 0 & 0 & 1 & 0 \\ 3 & -5 & 9 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccccccc} & & & ① & ② & ③ & ④ \end{array} \\ \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 & 5/12 & -1/3 \\ 0 & 1 & -3/2 & 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & 0 \end{bmatrix} \end{array}$$

— WE STILL END UP WITH A BASIS FOR $\mathbb{R}^4$, BUT DOESN'T EXTEND THE GIVEN COLLECTION; WHY? BECAUSE THE ORIGINAL COLLECTION WASN'T L.I. WE CAN ONLY EXTEND AN L.I. COLLECTION TO A BASIS; IF THE COLLECTION ISN'T L.I., WE'D HAVE TO REDUCE IT TO A L.I. COLLECTION FIRST, WHICH IS WHAT HAPPENED HERE (WE LOST THE THIRD VECTOR).