1. ($\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$: COLLECTION OF VECTORS IN SOME VECTOR SPACE $\mathcal{V}$)

CLAIM: $\mathcal{C}$ IS LINEARLY DEPENDENT IF, AND ONLY IF, ONE OF ITS VECTORS LIVES IN THE SPAN OF THE REST.

PROOF $\Rightarrow$: SUPPOSE THAT $\mathcal{C}$ IS LINEARLY DEPENDENT, I.E.:

$\exists$ SCALARS $a_1, a_2, \dots, a_n$ (NOT ALL ZERO) WITH $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$. (!)

TAKE SUCH a_i'S AND $\vec{v}_i$'S; BY HYPOTHESIS, AT LEAST ONE OF THE a_i'S IS NONZERO — WITHOUT LOSS OF GENERALITY, SUPPOSE THAT IT'S a_1; THEN $a_1 \neq 0$, SO WE CAN SCALE THE WHOLE EQUATION IN (!) BY $\frac{1}{a_1}$, OBTAINING $\vec{v}_1 + \frac{a_2}{a_1}\vec{v}_2 + \dots + \frac{a_n}{a_1}\vec{v}_n = \vec{0}$

$$\text{I.E., } \vec{v}_1 = \left(-\frac{a_2}{a_1}\right)\vec{v}_2 + \dots + \left(-\frac{a_n}{a_1}\right)\vec{v}_n$$

— BUT THE RIGHT-HAND SIDE IS A LINEAR COMBINATION OF $\{\vec{v}_2, \dots, \vec{v}_n\}$, AND THUS IS $\in \text{SPAN}\{\vec{v}_2, \dots, \vec{v}_n\}$, SO $\vec{v}_1 \in \text{SPAN}\{\vec{v}_2, \dots, \vec{v}_n\}$ AS WELL. ✓

PROOF $\Leftarrow$: SUPPOSE THAT ONE VECTOR OF $\mathcal{C}$ LIVES IN THE SPAN OF THE REST — WITHOUT LOSS OF GENERALITY, SUPPOSE IT'S $\vec{v}_1$; THEN

$$\vec{v}_1 \in \text{SPAN}\{\vec{v}_2, \dots, \vec{v}_n\}$$

SO, BY DEFINITION OF SPAN, $\exists$ SCALARS $a_2, \dots, a_n$ WITH

$$\vec{v}_1 = a_2\vec{v}_2 + \dots + a_n\vec{v}_n$$

BUT THEN $1\vec{v}_1 + (-a_2)\vec{v}_2 + \dots + (-a_n)\vec{v}_n = \vec{0}$

— BECAUSE THE COEFFICIENT OF $\vec{v}_1$ IS 1 ($\neq 0$), THIS IS A NONTRIVIAL LINEAR RELATION ON $\mathcal{C}$

SO, BY DEFINITION, $\mathcal{C}$ IS LINEARLY DEPENDENT ✓ ■

2. ($\mathcal{V}, \mathcal{W}$: FINITE COLLECTIONS OF VECTORS IN SOME VECTOR SPACE $\mathbb{V}$)

CLAIM: IF $\mathcal{V}$ IS L.I. IN $\mathbb{V}$ AND $\mathcal{W}$ SPANS $\mathbb{V}$, THEN $|\mathcal{V}| \leq |\mathcal{W}|$

PROOF: LET $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ AND $\mathcal{W} = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$.

STRATEGY: WHILE MAINTAINING A SPANNING SET, REPLACE THESE VECTORS, ONE AT A TIME, WITH RED VECTORS.

- STEP 1: $\vec{v}_1 \in \mathbb{V} = \text{SPAN}\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ (SINCE $\mathcal{W}$ SPANS $\mathbb{V}$), SO

  $\exists$ SCALARS $a_1, a_2, \dots, a_m$ SUCH THAT
  $\vec{v}_1 = a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 + \dots + a_m\vec{w}_m$.

  AT LEAST ONE a_i MUST BE NONZERO, BECAUSE OTHERWISE, THE RIGHT-HAND SIDE WOULD BE $\vec{0}$ AND WE'D OBTAIN A NONTRIVIAL LINEAR RELATION $1 \cdot \vec{v}_1 = \vec{0}$ ON $\mathcal{V}$, WHICH IS IMPOSSIBLE, SINCE $\mathcal{V}$ IS LINEARLY INDEPENDENT.

  WITHOUT LOSS OF GENERALITY, SUPPOSE THAT IT'S a_1 (IF NOT, REORDER THE VECTORS OF $\mathcal{W}$ SO THAT IT IS).

  THIS MEANS THAT WE CAN REPLACE $\vec{w}_1$ BY $\vec{v}_1$ IN OUR COLLECTION WHILE PRESERVING ITS SPAN.

  NET EFFECT: $\{\vec{v}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_m\}$ SPANS $\mathbb{V}$.

- STEP 2: $\vec{v}_2 \in \mathbb{V} = \text{SPAN}\{\vec{v}_1, \vec{w}_2, \vec{w}_3, \dots, \vec{w}_m\}$, SO

  $\exists$ SCALARS $a_1, a_2, \dots, a_m$ SUCH THAT
  $\vec{v}_2 = a_1\vec{v}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 + \dots + a_m\vec{w}_m$.

  AT LEAST ONE OF THE "BLUE" a_i'S MUST BE NONZERO, BECAUSE OTHERWISE, $\vec{v}_2 = a_1\vec{v}_1$ GIVES US A NONTRIVIAL LINEAR RELATION $1 \cdot \vec{v}_2 + (-a_1)\vec{v}_1 = \vec{0}$ ON $\mathcal{V}$ (IMPOSSIBLE).

  AGAIN, WITHOUT LOSS OF GENERALITY, SUPPOSE IT'S a_2. THEN WE CAN REPLACE $\vec{w}_2$ BY $\vec{v}_2$ IN OUR COLLECTION WHILE PRESERVING ITS SPAN.

  NET EFFECT: $\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \dots, \vec{w}_m\}$ SPANS $\mathbb{V}$.

- WE CAN CONTINUE IN THIS FASHION WITH EACH OF $\vec{v}_3, \dots, \vec{v}_n$ IN TURN — THE IMPORTANT POINT BEING THAT IN THE END, WE END UP WITH A SPANNING SET WITH m VECTORS THAT CONTAINS ALL n OF THE VECTORS $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. THUS, $n \leq m$, I.E., $|\mathcal{V}| \leq |\mathcal{W}|$ ■

3. (a) NOPE!
   (b) NOPE!
   (c) IF A COLLECTION $\mathcal{C}$ BOTH SPANS V AND IS L.I. IN V, WE CALL IT A BASIS FOR V

(THIS SPECIAL CONFLUENCE OF PROPERTIES IS IMPORTANT ENOUGH TO WARRANT A SPECIAL NAME — WE'LL HAVE A LOT MORE TO SAY ABOUT BASES AS WE GO ON.)

4. GIVEN A COLLECTION $\mathcal{C}$ OF COLUMN VECTORS IN $\mathbb{R}^m$, WE CAN FIND A BASIS FOR SPAN($\mathcal{C}$) COMPUTATIONALLY BY REDUCING THE MATRIX $[\mathcal{C}]$ AND LOCATING ITS PIVOT COLUMNS — THE CORRESPONDING VECTORS OF $\mathcal{C}$ ARE THE DESIRED BASIS.

→* BE SURE TO TAKE THE VECTORS FROM THE ORIGINAL COLLECTION $\mathcal{C}$, AND NOT THE REDUCED MATRIX! REDUCING THE MATRIX JUST TELLS US WHICH ONES TO TAKE.

WHY DO THESE VECTORS DO THE JOB?

IF WE WERE TO START OVER WITH A MULTIPLY-AUGMENTED SYSTEM WITH OUR BASIS ON THE LEFT AND THE REST OF $\mathcal{C}$ ON THE RIGHT, WE'D SEE THAT:

- OUR CHOSEN VECTORS ARE L.I. (EACH LEFT-SIDE COLUMN HAS A PIVOT!)

AND • EACH OF THE REST OF THE VECTORS IS AN L.C. OF THE CHOSEN VECTORS, SO THE REST CAN BE REMOVED FROM $\mathcal{C}$ WITHOUT CHANGING ITS SPAN. (EACH AUGMENTED COLUMN GIVES A CONSISTENT SYSTEM)

E.G., $[\mathcal{C}] \Rightarrow [\mathcal{C}_{\text{PIVOTS}} \mid \mathcal{C}_{\text{FREE}}]$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. CLAIM: IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SPANS V AND $\vec{v}_4 \in V$, THEN $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ IS LINEARLY DEPENDENT.

PROOF: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SPANS V AND $\vec{v}_4 \in V$; THEN $\vec{v}_4 \in V = \text{SPAN}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SO BY DEFINITION, $\exists$ SCALARS $\alpha_1, \alpha_2, \alpha_3$ SUCH THAT $\vec{v}_4 = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$.
WELL, THEN $1 \cdot \vec{v}_4 + (-\alpha_1)\vec{v}_1 + (-\alpha_2)\vec{v}_2 + (-\alpha_3)\vec{v}_3 = \vec{0}$, WHICH IS A NONTRIVIAL LINEAR RELATION ON $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$.
SO, BY DEFINITION, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ IS LINEARLY DEPENDENT. ■

6. CLAIM: IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ IS L.I. IN V, THEN $\{\vec{v}_1, \vec{v}_2\}$ DOES NOT SPAN V.

PROOF: (BECAUSE OUR CONCLUSION IS A NEGATIVE STATEMENT, WE MIGHT BE BEST OFF USING PROOF BY CONTRADICTION: WE CAN SHOW $P \Rightarrow Q$ BY PROVING THAT [P AND NOT Q] IS FALSE)

SUPPOSE THAT $\overbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ IS L.I. IN } V}^{(P)}$
AND THAT $\underbrace{\{\vec{v}_1, \vec{v}_2\} \text{ SPANS } V}_{(\text{NOT } Q)}$

THEN ① $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{0} \Rightarrow \alpha_1, \alpha_2, \alpha_3 = 0$
AND ② $\forall\, \vec{v} \in V$, $\exists$ SCALARS β_1, β_2 WITH $\vec{v} = \beta_1\vec{v}_1 + \beta_2\vec{v}_2$.

(WHAT TO DO? WELL, $\vec{v}_3$ IS THE DISTINGUISHED VECTOR THAT WE THREW OUT — HOW ABOUT USING IT IN ②?)

$\vec{v}_3 \in V$, SO BY ②, $\exists$ SCALARS β_1, β_2 WITH $\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2$.

(IS ① HAPPY WITH THIS? NO...)

NONSENSE!

THUS, $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + (-1)\vec{v}_3 = 0$, SO ① $\Rightarrow \beta_1, \beta_2, -1 = 0$

THIS IS FALSE, SO WE'VE SHOWN OUR IMPLICATION VIA CONTRADICTION ■

7. ($\mathcal{C}, \mathcal{D}$: COLLECTIONS OF VECTORS IN A VECTOR SPACE $\mathcal{V}$, WITH $\mathcal{C} \subset \mathcal{D}$)

(a) IF $\mathcal{C}$ SPANS $\mathcal{V}$ AND $\mathcal{C} \subset \mathcal{D}$, THEN $\mathcal{D}$ ALSO SPANS $\mathcal{V}$.

[WHY? IF $\mathcal{C}$ SPANS $\mathcal{V}$, WE CAN INSERT ANY VECTORS IN ITS SPAN — I.E., ANY VECTORS IN $\mathcal{V}$ — TO IT WITHOUT CHANGING ITS SPAN; IN PARTICULAR, IF WE INSERT THE REST OF $\mathcal{D}$ INTO IT, THE RESULTING COLLECTION ($\mathcal{D}$) SPANS $\mathcal{V}$ AS WELL.

(b) IF $\mathcal{C}$ IS L.I. AND $\mathcal{C} \subset \mathcal{D}$, NOTHING CAN BE CONCLUDED ABOUT $\mathcal{D}$.

[WHY? WELL, $\mathcal{D}$ COULD CONTAIN VECTORS ALREADY IN THE SPAN OF $\mathcal{C}$ — EVEN THE ZERO VECTOR — WHICH WOULD KILL LINEAR INDEPENDENCE. ON THE OTHER HAND, $\mathcal{D}$ COULD ALSO BE L.I. AS WELL!

(c) IF $\mathcal{D}$ SPANS $\mathcal{V}$ AND $\mathcal{C} \subset \mathcal{D}$, NOTHING CAN BE CONCLUDED ABOUT $\mathcal{C}$.

[WHY? WELL, CONSIDER THE EXTREME CASE $\mathcal{C} = \emptyset$: THEN SPAN($\mathcal{C}$) WOULD BE $\{\vec{0}\}$, WHICH CERTAINLY MIGHT NOT BE ALL OF $\mathcal{V}$!

(d) IF $\mathcal{D}$ IS L.I. AND $\mathcal{C} \subset \mathcal{D}$, THEN $\mathcal{C}$ IS ALSO L.I.

[WHY? WE CAN REMOVE ANY VECTOR(S) WE LIKE FROM A L.I. COLLECTION, AND IT STAYS L.I.

8. $|\mathcal{C}_{L.I.}| \le |\mathcal{C}_{BASIS}| \le |\mathcal{C}_{SPAN}|$

BECAUSE $\mathcal{C}_{BASIS}$ IS L.I. + $\mathcal{C}_{SPAN}$ SPANS.

BECAUSE $\mathcal{C}_{L.I.}$ IS L.I. AND $\mathcal{C}_{BASIS}$ SPANS!

9. $$\begin{bmatrix} 1 & 3 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ -1 & -4 & 3 & 2 \\ 1 & 0 & 1 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 2 \\ 6 \end{bmatrix} \right\}$ IS A BASIS FOR THE SPAN

10. $$\begin{bmatrix} -1 & 1 & 3 \\ -2 & 3 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ IS A BASIS FOR THE SPAN

11. $$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & 1 \\ 2 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} \right\}$ IS A BASIS FOR THE SPAN.