1. A **linear relation on** \( C \) **is a way of writing** \( \vec{0} \) **as a linear combination of** \( \vec{v} \)
   
   \[ \text{I.e., a choice of scalars } a_1, a_2, \ldots, a_n \in F \text{ for which } a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}. \]

   • The **trivial linear relation on a collection** is the one for which all chosen scalars are zero.

   (Such a linear relation tells us nothing about \( C \), because we can obviously get \( \vec{0} \) by taking our coefficients to all be zero!)

2. \( C \) is **linearly independent** if the only linear relation on \( C \) is the trivial one; i.e., if

   \[ a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0} \Rightarrow a_1, a_2, \ldots, a_n = 0. \]

   A priori, this tells us that the zero vector can only be written as a L.C. of \( C \) in the "trivial" way; but this tells us that any vector \( \vec{v} \in \mathbb{V} \) can be written as a L.C. of \( C \) in at most one way, too:

   If \( a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{v} = b_1 \vec{v}_1 + b_2 \vec{v}_2 + \cdots + b_n \vec{v}_n \),

   then \( (a_1 - b_1) \vec{v}_1 + (a_2 - b_2) \vec{v}_2 + \cdots + (a_n - b_n) \vec{v}_n = \vec{0} \)

   \[ \Rightarrow a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n = 0 \] (by linear independence)

   \[ \Rightarrow a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n. \]

   In summary, any two ways of writing \( \vec{v} \) as a L.C. of \( C \) must have exactly the same coefficients—so there can be at most one way of doing so.

   **Linear independence of** \( C \) **does not say anything at all about whether or not a vector can be written as a L.C. of** \( C \) **— it merely says that if one can, the way of doing so is unique!**

3. \( C \) is linearly dependent, then, if there is a nontrivial linear relation on \( C \), i.e., if

   \[ \exists a_1, a_2, \ldots, a_n \in F \text{ not all zero such that } a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n = \vec{0}. \]

   * Note that this is simply the logical negation of the definition of \( C \) being linearly independent—i.e., the existence of a counterexample to that implication.

4. While preserving the linear independence of a collection \( C \), we can:

   * **Insert** any vector that is not a linear combination of those already present in the collection.
   * **Remove** any vector in the collection.
   * **Replace** any vector \( \vec{v} \) in \( C \) by any linear combination of \( C \) having a nonzero coefficient of \( \vec{v} \).

   E.g., if \( \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \) is L.I.,

   then so is \( \{4\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3, \vec{v}_1, \vec{v}_2, \vec{v}_3\} \) (nonzero coefficient of \( \vec{v}_3 \))

   (We can prove these assertions via the)

   (Definition of linear independence)
5. (c: finite collection of column vectors in $\mathbb{R}^m$)

(a) If $C$ is linearly independent, all systems arising from $C$
    have either a unique solution or no solutions, because
    as discussed in Problem 2, every vector in $\mathbb{R}^m$
    can be written as an L.C. of $C$ in at most one way.

(b) If, on the other hand, $C$ is linearly dependent, then
    all systems arising from $C$ have either infinitely
    many solutions or no solutions.

6. (a) Any collection $C$ containing the zero vector is
    linearly dependent:

    Suppose that $\vec{0} \in C$; then we automatically have
    a nontrivial linear relation on $C$: $1 \cdot \vec{0} = \vec{0}$.

    So by definition, $C$ is linearly dependent. ■

(b) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then one of
    the three vectors must be a l.c. of the other two:

    Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

    Then $\exists$ scalars $\alpha_1, \alpha_2, \alpha_3$, not all zero,
    with $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 = \vec{0}$.

    If $\alpha_1 \neq 0$, then $\vec{v}_1 = -\frac{\alpha_2}{\alpha_1} \vec{v}_2 - \frac{\alpha_3}{\alpha_1} \vec{v}_3,\text{ so } \vec{v}_1 = -\frac{\alpha_2}{\alpha_1} \vec{v}_2 - \frac{\alpha_3}{\alpha_1} \vec{v}_3$.

    Thus, $\vec{v}_1$ is a l.c. of $\{\vec{v}_2, \vec{v}_3\}$. ■

    If $\alpha_2 \neq 0$, then $\vec{v}_2 = -\frac{\alpha_1}{\alpha_2} \vec{v}_1 - \frac{\alpha_3}{\alpha_2} \vec{v}_3,\text{ so } \vec{v}_2 = -\frac{\alpha_1}{\alpha_2} \vec{v}_1 - \frac{\alpha_3}{\alpha_2} \vec{v}_3$.

    Thus, $\vec{v}_2$ is a l.c. of $\{\vec{v}_1, \vec{v}_3\}$. ■

    If $\alpha_3 \neq 0$, then $\vec{v}_3 = -\frac{\alpha_1}{\alpha_3} \vec{v}_1 - \frac{\alpha_2}{\alpha_3} \vec{v}_2,\text{ so } \vec{v}_3 = -\frac{\alpha_1}{\alpha_3} \vec{v}_1 - \frac{\alpha_2}{\alpha_3} \vec{v}_2$.

    Thus, $\vec{v}_3$ is a l.c. of $\{\vec{v}_1, \vec{v}_2\}$. ■
7.a) If \( \mathbf{c} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is L.I., then \( \mathbf{c} = \{\mathbf{v}_1 - 4\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_3\} \) is also L.I.:

- **ON PRINCIPLE:** All that we've done is replaced one vector \((\mathbf{v}_2)\) of \( \mathbf{c} \) with a linear combination of \( \mathbf{v}_1 \) having a nonzero coefficient for \( \mathbf{v}_1 \), which is one of our L.I.-preserving operations.

- **DIRECTLY:** Suppose that \( \mathbf{c} \) is linearly independent, i.e.,
  \[
  \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0} \implies \alpha_1, \alpha_2, \alpha_3 = 0 (\dagger)
  \]
  
  To show that \( \mathbf{c}' \) is linearly independent, suppose that
  \[
  \beta_1 (\mathbf{v}_1 - 4\mathbf{v}_2) + \beta_2 \mathbf{v}_2 + \beta_3 \mathbf{v}_3 = \mathbf{0}, \quad \text{i.e.,}
  \]
  \[
  \beta_1 \mathbf{v}_1 + (-4\beta_1 + \beta_2) \mathbf{v}_2 + \beta_3 \mathbf{v}_3 = \mathbf{0}
  \]
  (Now take \( \alpha_1 = \beta_1, \alpha_2 = -4\beta_1 + \beta_2, \) and \( \alpha_3 = \beta_3 \) in \((\dagger))
  
  Which \( \implies \beta_1 = 0, -4\beta_1 + \beta_2 = 0, \) and \( \beta_3 = 0 \).
  
  Thus \( \beta_1, \beta_2, \beta_3 = 0 \), and we've verified that \( \mathbf{c}' \) is linearly independent. \( \blacksquare \)

7.b) If \( \mathbf{c} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is L.I., then so is \( \mathbf{c}' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \):

- **ON PRINCIPLE:** \( \mathbf{c}' \) is just \( \mathbf{c} \) with one vector removed—this is one of our operations that preserves linear independence. \( \blacksquare \)

- **DIRECTLY:** Suppose that \( \mathbf{c} \) is linearly independent, i.e.,
  \[
  \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = \mathbf{0} \implies \alpha_1, \alpha_2, \alpha_3 = 0 (\dagger)
  \]
  
  To show that \( \mathbf{c}' \) is linearly independent, suppose that
  \[
  \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \beta_3 \mathbf{v}_3 = \mathbf{0}
  \]
  Then
  \[
  \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \alpha \mathbf{v}_3 = \mathbf{0}
  \]
  (Now take \( \alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \alpha \) in \((\dagger))
  
  Which \( \implies \beta_1 = 0, \beta_2 = 0 \) and \( \beta_3 = 0 \) by hypothesis, so \( \beta_1, \beta_2 = 0 \). \( \blacksquare \)

7.c) If \( \mathbf{c} = \{\mathbf{v}_1, \mathbf{w}_2\} \) is L.I., then \( \mathbf{c}' = \{\mathbf{v}_1 - \mathbf{w}_2, \mathbf{v}_2 + 3\mathbf{w}_2\} \) is L.I.:

- **ON PRINCIPLE:** Using the fact that while preserving linear independence, we can replace any vector of the collection with any L.I. of the collection having a nonzero coefficient for it:
  \[
  \{\mathbf{v}_1, \mathbf{w}_2\} \text{ L.I. } \Rightarrow \{\mathbf{v}_1 - \mathbf{w}_2, \mathbf{w}_2\} \text{ L.I.}
  \]
  
  Now, \( \mathbf{v} + 3\mathbf{w} = \frac{1}{2} (2\mathbf{v} - \mathbf{w}) + \frac{3}{2} \mathbf{w} \),
  
  So replacing \( \mathbf{w} \), \( \{\mathbf{v}_1 - \mathbf{w}_2, \mathbf{v}_2 + 3\mathbf{w}_2\} \) is L.I. \( \blacksquare \)

- **DIRECTLY:** We know that since \( \mathbf{c} \) is L.I.,
  \[
  \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{w}_2 = \mathbf{0} \implies \alpha_1, \alpha_2 = 0 (\dagger)
  \]
  Suppose that
  \[
  \beta_1 (2\mathbf{v} - \mathbf{w}) + \beta_2 (\mathbf{v} + 3\mathbf{w}) = \mathbf{0}, \quad \text{i.e.,}
  \]
  \[
  (2\beta_1 + \beta_2) \mathbf{v} + (-\beta_1 + 3\beta_2) \mathbf{w} = \mathbf{0}
  \]
  (Now take \( \alpha_1 = 2\beta_1 + \beta_2, \alpha_2 = -\beta_1 + 3\beta_2 \) in \((\dagger))
  
  Which \( \implies 2\beta_1 + \beta_2 = 0 \) and \( -\beta_1 + 3\beta_2 = 0 \)
  
  Quickly solving, \[
  \begin{bmatrix}
  2 & 1 \\
  -1 & 3
  \end{bmatrix}
  \begin{bmatrix}
  \beta_1 \\
  \beta_2
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  0
  \end{bmatrix}
  \]
  So \( \beta_1 = 0, \beta_2 = 0 \) \( \blacksquare \)
(Recall that a collection \( C \) is l.i. if after reducing \( [C] \), every column has a pivot.)

8. \[
\begin{bmatrix}
1 & -1 & 2 \\
2 & 0 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\color{blue}{1} & 0 & 0 \\
0 & \color{blue}{1} & 0 \\
0 & 0 & \color{blue}{1} \\
\end{bmatrix}
\]

Yes, these are l.i.

9. \[
\begin{bmatrix}
4 & -4 & -8 \\
-12 & 13 & 27 \\
-8 & 7 & 13 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\color{blue}{1} & 0 & 1 \\
0 & \color{blue}{1} & 3 \\
0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\color{blue}{1} & 0 & 0 \\
0 & \color{blue}{1} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\( \alpha_1 = -9 \gamma_3 \)
\( \alpha_2 = -3 \gamma_3 \)

\( \Rightarrow \) Not l.i.

To find a nontrivial linear relation, just take \( \gamma_3 \) to be anything nonzero (e.g., \( \gamma_3 = 1 \)) to find coefficients:

\( \alpha_3 = 1 \Rightarrow \alpha_1 = -1, \alpha_2 = -3 \)

\( \therefore \)

\[
\begin{bmatrix}
-4 \\
-12 \\
-8 \\
\end{bmatrix}
- \begin{bmatrix}
-4 \\
13 \\
7 \\
\end{bmatrix}
+ \begin{bmatrix}
8 \\
27 \\
13 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(Nontrivial linear relation)

10. (Like Problem 9, but with a "c"... just follow with it!)

\[
\begin{bmatrix}
1 & 2 & 0 \\
-1 & 0 & 3 \\
2 & 1 & c \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\color{blue}{1} & 2 & 0 \\
0 & 2 & 3 \\
0 & -3 & c \\
\end{bmatrix}.
\]

\( \rightarrow \)

\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\color{blue}{1} & 0 & -3 \\
0 & \color{blue}{1} & 3 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

This will be a pivot, unless \( c + 3 = 0 \)... so the vectors will be linearly dependent if \( c = -9 \). If \( c = -9 \), this becomes

\[
\begin{bmatrix}
\color{blue}{1} & 0 & -3 \\
0 & \color{blue}{1} & 3 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\( \Rightarrow \) \( \alpha_1 = -9\gamma_3, \alpha_2 = -2\gamma_3 \)
\( \alpha_3: \text{free} \)

11. (One more time, now with three unknowns!)

\[
\begin{bmatrix}
1 & o & a \\
0 & 1 & b \\
1 & 1 & c \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & c-a-b \\
\end{bmatrix}
\]

We'll get our third pivot here if, and only if, \( c-a-b \neq 0 \).

\( \therefore \) These column vectors will be l.i. if \( c-a-b \neq 0 \); they will be linearly dependent if \( c-a-b = 0 \).