(V: VECTOR SPACE OVER A FIELD F;
$\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$: FINITE COLLECTION OF VECTORS IN V)

1. A LINEAR RELATION ON $\mathcal{C}$ IS A WAY OF WRITING $\vec{0}$ AS A LINEAR COMBINATION OF $\mathcal{C}$

   — I.E., A CHOICE OF SCALARS $a_1, a_2, \ldots, a_n \in F$ FOR WHICH $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}$.

   - THE TRIVIAL LINEAR RELATION ON A COLLECTION IS THE ONE FOR WHICH ALL CHOSEN SCALARS ARE ZERO.

   (SUCH A LINEAR RELATION TELLS US NOTHING ABOUT $\mathcal{C}$, BECAUSE WE CAN OBVIOUSLY GET $\vec{0}$ BY TAKING OUR COEFFICIENTS TO ALL BE ZERO!)

2. $\mathcal{C}$ IS LINEARLY INDEPENDENT IF THE ONLY LINEAR RELATION ON $\mathcal{C}$ IS THE TRIVIAL ONE, I.E., IF

$$\boxed{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0} \implies a_1, a_2, \ldots, a_n = 0}$$

A PRIORI, THIS TELLS US THAT THE ZERO VECTOR CAN ONLY BE WRITTEN AS A L.C. OF $\mathcal{C}$ IN THE "TRIVIAL" WAY; BUT THIS TELLS US THAT ANY VECTOR $\vec{v} \in V$ CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$ IN AT MOST ONE WAY, TOO: !

IF $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{v} = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \cdots + \beta_n\vec{v}_n$,

THEN $(a_1 - \beta_1)\vec{v}_1 + (a_2 - \beta_2)\vec{v}_2 + \cdots + (a_n - \beta_n)\vec{v}_n = \vec{0}$

$\implies a_1 - \beta_1,\ a_2 - \beta_2,\ \ldots,\ a_n - \beta_n = 0$ (BY LINEAR INDEPENDENCE)

$\implies a_1 = \beta_1,\ a_2 = \beta_2,\ \ldots,\ a_n = \beta_n$

IN SUMMARY, ANY TWO WAYS OF WRITING $\vec{v}$ AS A L.C. OF $\mathcal{C}$ MUST HAVE EXACTLY THE SAME COEFFICIENTS — SO THERE CAN BE AT MOST ONE WAY OF DOING SO.

* LINEAR INDEPENDENCE OF $\mathcal{C}$ DOESN'T SAY ANYTHING AT ALL ABOUT WHETHER OR NOT A VECTOR CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$ — IT MERELY SAYS THAT IF ONE CAN, THE WAY OF DOING SO IS UNIQUE!

3. $\mathcal{C}$ IS LINEARLY DEPENDENT, THEN, IF THERE IS A NONTRIVIAL LINEAR RELATION ON $\mathcal{C}$, I.E., IF

$$\boxed{\exists\, a_1, a_2, \ldots, a_n \in F \text{ NOT ALL ZERO SUCH THAT } a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \vec{0}}$$

* NOTE THAT THIS IS SIMPLY THE LOGICAL NEGATION OF THE DEFINITION OF $\mathcal{C}$ BEING LINEARLY INDEPENDENT — I.E., THE EXISTENCE OF A COUNTEREXAMPLE TO THAT IMPLICATION.

4. WHILE PRESERVING THE LINEAR INDEPENDENCE OF A COLLECTION $\mathcal{C}$, WE CAN:

   - INSERT ANY VECTOR THAT IS NOT A LINEAR COMBINATION OF THOSE ALREADY PRESENT IN THE COLLECTION.
   - REMOVE ANY VECTOR IN THE COLLECTION.
   - REPLACE ANY VECTOR $\vec{v}$ IN $\mathcal{C}$ BY ANY L.C. OF $\mathcal{C}$ HAVING A NONZERO COEFFICIENT OF $\vec{v}$.

     E.G., IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ IS L.I., THEN SO IS $\{4\vec{v}_1 + \vec{v}_2 - 3\vec{v}_3, \vec{v}_2, \vec{v}_3\}$

     NONZERO COEFFICIENT OF $\vec{v}_1$

(WE CAN PROVE THESE ASSERTIONS VIA THE DEFINITION OF LINEAR INDEPENDENCE)

5. ($\mathcal{C}$: FINITE COLLECTION OF COLUMN VECTORS IN $\mathbb{R}^m$)

(a) IF $\mathcal{C}$ IS LINEARLY INDEPENDENT, ALL SYSTEMS ARISING FROM $\mathcal{C}$ HAVE EITHER A UNIQUE SOLUTION OR NO SOLUTIONS, BECAUSE AS DISCUSSED IN PROBLEM 2, EVERY VECTOR IN $\mathbb{R}^m$ CAN BE WRITTEN AS AN L.C. OF $\mathcal{C}$ IN AT MOST ONE WAY.

(IF, ON THE OTHER HAND, $\mathcal{C}$ IS LINEARLY DEPENDENT, THEN ALL SYSTEMS ARISING FROM $\mathcal{C}$ HAVE EITHER INFINITELY MANY SOLUTIONS OR NO SOLUTIONS.)

(b) TO CHECK WHETHER A COLLECTION $\mathcal{C}$ OF COLUMN VECTORS IS LINEARLY INDEPENDENT, WE JUST FORM THE MATRIX $[\mathcal{C}]$ (UNAUGMENTED) AND REDUCE IT — IF THERE IS A PIVOT IN EVERY COLUMN, THEN $\mathcal{C}$ IS LINEARLY INDEPENDENT.

WHY? WHEN WE USE $[\mathcal{C}]$ HERE, WE'RE JUST LEAVING OFF AN AUGMENTATION BY THE ZERO VECTOR, AND THUS SOLVING FOR $\vec{0}$ AS A L.C. OF THESE VECTORS — IF EVERY COLUMN HAS A PIVOT, THEN WHEN WE SOLVE FOR THE COEFFICIENTS, WE FIND THAT THEY ALL MUST EQUAL ZERO (I.E., THE ONLY SOLUTION IS THE TRIVIAL L.C.; IF NOT, THEN THE (NON-PIVOT) FREE VARIABLE(S) GIVE US A NONTRIVIAL LINEAR RELATION.

6. (a) (ANY COLLECTION $\mathcal{C}$ CONTAINING THE ZERO VECTOR IS LINEARLY DEPENDENT:)

SUPPOSE THAT $\vec{0} \in \mathcal{C}$; THEN WE AUTOMATICALLY HAVE A NONTRIVIAL LINEAR RELATION ON $\mathcal{C}$: $\underset{\uparrow\ \text{NONZERO COEFFICIENT}}{1}\cdot\underset{\uparrow\ \text{VECTOR OF }\mathcal{C}}{\vec{0}} = \vec{0}$,

SO BY DEFINITION, $\mathcal{C}$ IS LINEARLY DEPENDENT. ■

(b) (IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ IS LINEARLY DEPENDENT, THEN ONE OF THE THREE VECTORS MUST BE A L.C. OF THE OTHER TWO:)

SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ IS LINEARLY DEPENDENT.

THEN $\exists$ SCALARS a_1, a_2, a_3, NOT ALL ZERO, WITH $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$.

IF $a_1 \neq 0$, THEN $a_1\vec{v}_1 = -a_2\vec{v}_2 - a_3\vec{v}_3$, SO $\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2 - \frac{a_3}{a_1}\vec{v}_3$;
THUS, $\vec{v}_1$ IS A L.C. OF $\{\vec{v}_2, \vec{v}_3\}$ ✓

IF $a_2 \neq 0$, THEN $a_2\vec{v}_2 = -a_1\vec{v}_1 - a_3\vec{v}_3$, SO $\vec{v}_2 = -\frac{a_1}{a_2}\vec{v}_1 - \frac{a_3}{a_2}\vec{v}_3$;
THUS, $\vec{v}_2$ IS A L.C. OF $\{\vec{v}_1, \vec{v}_3\}$ ✓

IF $a_3 \neq 0$, THEN $a_3\vec{v}_3 = -a_1\vec{v}_1 - a_2\vec{v}_2$, SO $\vec{v}_3 = -\frac{a_1}{a_3}\vec{v}_1 - \frac{a_2}{a_3}\vec{v}_2$;
THUS, $\vec{v}_3$ IS A L.C. OF $\{\vec{v}_1, \vec{v}_2\}$ ✓ ■

JUST LIKE THE FIRST CASE

7. (a) IF $\mathcal{C}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ IS L.I., THEN $\mathcal{C}'=\{\vec{v}_1-4\vec{v}_2,\vec{v}_2,\vec{v}_3\}$ IS ALSO L.I.:

- ON PRINCIPLE: ALL THAT WE'VE DONE IS REPLACED ONE VECTOR ($\vec{v}_1$) OF $\mathcal{C}$ WITH A LINEAR COMBINATION OF $\mathcal{C}$ HAVING A NONZERO COEFFICIENT FOR $\vec{v}_1$ — WHICH IS ONE OF OUR L.I.-PRESERVING OPERATIONS ■

- DIRECTLY: SUPPOSE THAT $\mathcal{C}$ IS LINEARLY INDEPENDENT, I.E.,

$$\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3=\vec{0} \implies \alpha_1,\alpha_2,\alpha_3=0 \quad (!)$$

TO SHOW THAT $\mathcal{C}'$ IS LINEARLY INDEPENDENT, SUPPOSE THAT

$$\beta_1(\vec{v}_1-4\vec{v}_2)+\beta_2\vec{v}_2+\beta_3\vec{v}_3=\vec{0}, \quad \text{I.E.,}$$

$$\beta_1\vec{v}_1+(-4\beta_1+\beta_2)\vec{v}_2+\beta_3\vec{v}_3=\vec{0}$$

(NOW TAKE $\alpha_1=\beta_1$, $\alpha_2=-4\beta_1+\beta_2$, AND $\alpha_3=\beta_3$ IN (!))

WHICH $\Rightarrow$ $\beta_1=0$, $-4\beta_1+\beta_2=0$, AND $\beta_3=0$.

$\beta_2=-4\beta_1=0$

THUS $\beta_1,\beta_2,\beta_3=0$, AND WE'VE VERIFIED THAT $\mathcal{C}'$ IS LINEARLY INDEPENDENT. ■

(b) IF $\mathcal{C}=\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ IS L.I., THEN SO IS $\mathcal{C}'=\{\vec{v}_1,\vec{v}_2\}$:

- ON PRINCIPLE: $\mathcal{C}'$ IS JUST $\mathcal{C}$ WITH ONE VECTOR REMOVED — THIS IS ONE OF OUR OPERATIONS THAT PRESERVES LINEAR INDEPENDENCE ■

- DIRECTLY: SUPPOSE THAT $\mathcal{C}$ IS LINEARLY INDEPENDENT, I.E.,

$$\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3=\vec{0} \implies \alpha_1,\alpha_2,\alpha_3=0 \quad (!)$$

TO SHOW THAT $\mathcal{C}'$ IS LINEARLY INDEPENDENT, SUPPOSE THAT $\beta_1\vec{v}_1+\beta_2\vec{v}_2=\vec{0}$.

THEN $\beta_1\vec{v}_1+\beta_2\vec{v}_2+0\vec{v}_3=\vec{0}$

(NOW TAKE $\alpha_1=\beta_1$, $\alpha_2=\beta_2$, $\alpha_3=0$ IN (!))

WHICH $\Rightarrow$ $\beta_1=0$, $\beta_2=0$, $0=0$ BY HYPOTHESIS, SO $\beta_1,\beta_2=0$ ■

(c) IF $\mathcal{C}=\{\vec{v},\vec{w}\}$ IS L.I., THEN $\mathcal{C}'=\{2\vec{v}-\vec{w},\vec{v}+3\vec{w}\}$ IS L.I.:

- ON PRINCIPLE: USING THE FACT THAT WHILE PRESERVING LINEAR INDEPENDENCE, WE CAN REPLACE ANY VECTOR OF THE COLLECTION WITH ANY L.C. OF THE COLLECTION HAVING A NONZERO COEFFICIENT FOR IT:

$$\{\vec{v},\vec{w}\} \text{ L.I.} \Rightarrow \{2\vec{v}-\vec{w},\vec{w}\} \text{ L.I.}$$

NOW, $\vec{v}+3\vec{w}=\frac{1}{2}(2\vec{v}-\vec{w})+\frac{7}{2}\vec{w}$,

NONZERO COEFFICIENT

SO REPLACING $\vec{w}$, $\{2\vec{v}-\vec{w},\vec{v}+3\vec{w}\}$ IS L.I. ■

- DIRECTLY: WE KNOW THAT SINCE $\mathcal{C}$ IS L.I.,

$$\alpha_1\vec{v}+\alpha_2\vec{w}=\vec{0} \implies \alpha_1,\alpha_2=0 \quad (!)$$

SUPPOSE THAT $\beta_1(2\vec{v}-\vec{w})+\beta_2(\vec{v}+3\vec{w})=\vec{0}$, I.E.,

$$(2\beta_1+\beta_2)\vec{v}+(-\beta_1+3\beta_2)\vec{w}=\vec{0},$$

(NOW TAKE $\alpha_1=2\beta_1+\beta_2$, $\alpha_2=-\beta_1+3\beta_2$ IN (!))

WHICH $\Rightarrow$ $2\beta_1+\beta_2=0$ AND $-\beta_1+3\beta_2=0$

QUICKLY SOLVING, $\begin{array}{cc} \beta_1 & \beta_2 \end{array}$ $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 3 & 0 \end{array}\right] \rightsquigarrow \cdots \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$,

SO $\beta_1=0$, $\beta_2=0$ ■

(RECALL THAT A COLLECTION $\mathcal{C}$ IS L.I. $\Leftrightarrow$ AFTER REDUCING $[\mathcal{C}]$, EVERY COLUMN HAS A PIVOT)

8. $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ YES, THESE ARE L.I. ✓

REMEMBER THAT WHEN WE DO THIS, WE'RE JUST LEAVING OFF AN AUGMENTATION BY $\vec{0}$

9. $\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 4 & -4 & -8 \\ -12 & 13 & 27 \\ -8 & 7 & 13 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccc} & & \text{FREE} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow$ a_3 : FREE, $a_1 = -a_3$, $a_2 = -3a_3$

NOT L.I.

TO FIND A NONTRIVIAL LINEAR RELATION, JUST TAKE a_3 TO BE ANYTHING NONZERO (E.G., $a_3 = 1$) TO FIND COEFFICIENTS:

$a_3 = 1 \Rightarrow a_1 = -1,\ a_2 = -3$

$$\therefore\ -\begin{bmatrix} 4 \\ -12 \\ -8 \end{bmatrix} - 3\begin{bmatrix} -4 \\ 13 \\ 7 \end{bmatrix} + \begin{bmatrix} -8 \\ 27 \\ 13 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

NONTRIVIAL LINEAR RELATION!

10. (LIKE PROBLEM 9, BUT WITH A "C"... JUST ROLL WITH IT!)

$-2,\ +1$: $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \\ 2 & 1 & c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & -3 & c \end{bmatrix} \cdot \tfrac{1}{2}$

$\rightsquigarrow$ $-2,\ +3$: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3/2 \\ 0 & -3 & c \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & c + 9/2 \end{bmatrix}$

THIS WILL BE A PIVOT, UNLESS $c + 9/2 = 0$... SO THE VECTORS WILL BE LINEARLY DEPENDENT IF $c = -9/2$

IF $c = -9/2$, THIS BECOMES $\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \ (\text{FREE}) \end{array} \\ \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$, SO a_3 : FREE, $a_1 = 3a_3$, $a_2 = -\frac{3}{2}a_3$

TAKING $a_3 = 1$ (E.G.), WE GET $a_1 = 3$, $a_2 = -\frac{3}{2}$, AND THUS WE OBTAIN THE NONTRIVIAL LINEAR RELATION

$$3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -9/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

11. (ONE MORE TIME, NOW WITH THREE UNKNOWNS!)

-1: $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix} \rightsquigarrow$ -1: $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 1 & c-a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c-a-b \end{bmatrix}$

WE'LL GET OUR THIRD PIVOT HERE IF, AND ONLY IF, $c - a - b \neq 0$.

$\therefore$ THESE COLUMN VECTORS WILL BE L.I. IF $c - a - b \neq 0$; THEY WILL BE LINEARLY DEPENDENT IF $c - a - b = 0$.