1. THE TRANSPOSE OF A COLUMN VECTOR $\vec{x} \in \mathbb{R}^m$ IS A "ROW VECTOR" (OR A $1 \times m$ MATRIX) HAVING THE SAME ENTRIES.

$$\text{E.G.,} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

FOR $\vec{x}, \vec{y} \in \mathbb{R}^m$, WE CAN COMPUTE $\vec{x} \cdot \vec{y}$ AS $\vec{x}^T y$

$$\text{E.G.,} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 1 \cdot 2 + 2(-1) + 3 \cdot 2$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 1 \cdot 2 + 2(-1) + 3 \cdot 2$$

EQUAL!

2. A: $m \times n$ MATRIX.

(a) THE TRANSPOSE OF A, DENOTED $A^T$, IS THE $n \times m$ MATRIX WHOSE $j^{\text{TH}}$ ROW IS THE $j^{\text{TH}}$ COLUMN OF A; A TRANSPOSED MATRIX SHOULD BE VIEWED AS CONSISTING OF ROWS, RATHER THAN COLUMNS.

$$\text{E.G.,} \left[\begin{array}{c:c} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right]^T = \left[\begin{array}{ccc} 1 & 3 & 5 \\ \hdashline 2 & 4 & 6 \end{array}\right]$$

TRANSPOSING A MATRIX CAN ALSO BE VISUALIZED AS "FLIPPING" A MATRIX ABOUT ITS DIAGONAL.

(b) THE TRANSPOSE COMMUTES WITH ADDITION AND SCALING OF MATRICES: $(A+B)^T = A^T + B^T$, AND

$$(\alpha A)^T = \alpha A^T.$$

(c) THE TRANSPOSE OF A MATRIX COMPOSITION IS THE COMPOSITION OF THE TRANSPOSES IN THE OPPOSITE ORDER, I.E.,

$$(AB)^T = B^T A^T$$

(JUST AS WITH INVERSION)

3. IF A IS AN $m \times n$ MATRIX AND $\vec{y} \in \mathbb{R}^m$, THE ENTRIES OF $A^T \vec{y}$ ARE THE DOT PRODUCTS OF EACH COLUMN OF A WITH $\vec{y}$:

IF $A = \left[\vec{a}_1 \middle| \dots \middle| \vec{a}_n\right]$, THEN
$$A^T \vec{y} = \begin{bmatrix} \vec{a}_1 \\ \hline \vdots \\ \hline \vec{a}_n \end{bmatrix} \vec{y} = \begin{bmatrix} \vec{a}_1 \cdot \vec{y} \\ \vec{a}_2 \cdot \vec{y} \\ \vdots \\ \vec{a}_n \cdot \vec{y} \end{bmatrix}.$$

CLAIM: $\vec{y} \perp C(A) \Leftrightarrow A^T \vec{y} = \vec{0}$

PROOF: WRITE $A = \left[\vec{a}_1 \middle| \dots \middle| \vec{a}_n\right]$.

$\Rightarrow$: SUPPOSE $y \perp C(A)$, I.E., $\forall \vec{a} \in C(A)$, $\vec{y} \perp \vec{a}$. $(*)$

THEN $A^T \vec{y} = \begin{bmatrix} \vec{a}_1 \cdot \vec{y} \\ \vec{a}_2 \cdot \vec{y} \\ \vdots \\ \vec{a}_n \cdot \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ BY $(*)$ SINCE $a_i \in C(A)$

$= \vec{0}$ ✓

$\Leftarrow$: SUPPOSE $A^T \vec{y} = \vec{0}$, I.E., $\begin{bmatrix} \vec{a}_1 \cdot \vec{y} \\ \vec{a}_2 \cdot \vec{y} \\ \vdots \\ \vec{a}_n \cdot \vec{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

[ GOAL: $y \perp C(A)$, I.E., $\forall \vec{a} \in C(A)$, $\vec{y} \perp \vec{a}$
I.E., $\forall \vec{a} \in C(A)$, $\langle \vec{a}, \vec{y} \rangle = 0$.

LET $\vec{a} \in C(A)$ BE GIVEN.
THEN $\vec{a}$ IS A L.C. OF THE COLUMNS OF A, I.E.,

$$\vec{a} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_n \vec{a}_n.$$

THUS,
$$\begin{aligned} \langle \vec{a}, \vec{y} \rangle &= \langle \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_n \vec{a}_n, \vec{y} \rangle \\ &= \alpha_1 \langle \vec{a}_1, \vec{y} \rangle + \alpha_2 \langle \vec{a}_2, \vec{y} \rangle + \dots + \alpha_n \langle \vec{a}_n, \vec{y} \rangle \\ &= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \dots + \alpha_n \cdot 0 \\ &= 0, \end{aligned}$$
SO $\vec{a} \perp \vec{y}$ ✓

IN SUMMARY, $\vec{y} \perp C(A) \Leftrightarrow A^T \vec{y} = \vec{0}$. ■

4. IF $A$ AND $B$ ARE $m \times n$ MATRICES, THE ENTRIES OF $A^T B$ GIVE THE DOT PRODUCTS OF EACH VECTOR OF $A$ (IN THE ROWS) WITH EACH VECTOR OF $B$ (IN THE COLUMNS):

IF $A = \left[\vec{a}_1 \middle| \dots \middle| \vec{a}_n\right]$ AND $B = \left[\vec{b}_1 \middle| \dots \middle| \vec{b}_n\right]$,

THEN THE $j^{\text{TH}}$ COLUMN OF $A^T B$ IS $A^T(j^{\text{TH}}$ COLUMN OF $B) = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_j \\ \vec{a}_2 \cdot \vec{b}_j \\ \vdots \\ \vec{a}_n \cdot \vec{b}_j \end{bmatrix}$

PUTTING THESE TOGETHER, $A^T B = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 & \cdots & \vec{a}_1 \cdot \vec{b}_n \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 & \cdots & \vec{a}_2 \cdot \vec{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n \cdot \vec{b}_1 & \vec{a}_n \cdot \vec{b}_2 & \cdots & \vec{a}_n \cdot \vec{b}_n \end{bmatrix}$.

* * *

IF THE COLUMNS OF AN $m \times n$ MATRIX $A$ ARE ORTHONORMAL,

THEN $A^T A = \begin{bmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_n \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdots & \vec{a}_2 \cdot \vec{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n \cdot \vec{a}_1 & \vec{a}_n \cdot \vec{a}_2 & \cdots & \vec{a}_n \cdot \vec{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$

$A$: $m \times n$ MATRIX WITH LINEARLY INDEPENDENT COLUMNS.

5. $\underset{\mathbb{R}^n \to \mathbb{R}^m}{A}$ IS AN $m \times n$ MATRIX, SO $\underset{\mathbb{R}^m \to \mathbb{R}^n}{A^T}$ IS AN $n \times m$ MATRIX.

THUS $A^T A : \mathbb{R}^n \to \mathbb{R}^n$ IS A [SQUARE] $n \times n$ MATRIX.

CLAIM: $A^T A$ IS INJECTIVE (I.E., $N(A^T A) = \{\vec{0}\}$,
I.E., $A^T A \vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$)

PROOF: SUPPOSE THAT $A^T A \vec{x} = \vec{0}$.

THEN DOTTING WITH $\vec{x}$,
$$\begin{aligned} 0 = \vec{x} \cdot \vec{0} &= \vec{x} \cdot (A^T A \vec{x}) \\ &= \vec{x}^T A^T A \vec{x} \\ &= (A\vec{x})^T A\vec{x} \\ &= (A\vec{x}) \cdot (A\vec{x}) \end{aligned}$$

BUT $(A\vec{x}) \cdot (A\vec{x}) = 0 \Rightarrow A\vec{x} = \vec{0}$ ($\cdot$ IS POSITIVE-DEFINITE)

NOW, $A\vec{x} = \vec{0}$ MEANS THAT $\vec{x}$ GIVES COEFFICIENTS FOR A LINEAR RELATION ON THE COLUMNS OF $A$. BUT THE COLUMNS OF $A$ ARE LINEARLY INDEPENDENT, SO THE COEFFICIENTS MUST ALL BE 0, I.E., $\vec{x} = \vec{0}$.

IN SUMMARY, $A^T A \vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$, SO $A^T A$ IS INJECTIVE. ■

BY RANK+NULLITY, SINCE $A^T A : \mathbb{R}^n \to \mathbb{R}^n$ IS INJECTIVE, IT HAS NULLITY 0, SO ITS RANK IS $n$, I.E., $A^T A$ IS ALSO SURJECTIVE, AND THUS $A^T A$ IS BIJECTIVE, ∴ INVERTIBLE.

6. $P = A\,(A^TA)^{-1}\,A^T$

$\mathbb{R}^m \leftarrow \mathbb{R}^n \longleftarrow \mathbb{R}^n \leftarrow \mathbb{R}^m$

(a) $P: \mathbb{R}^m \to \mathbb{R}^m$ (DOMAIN + CODOMAIN ARE $\mathbb{R}^m$)

(b) CLAIM: IF $\vec{y} \in \mathbb{R}^m$ AND $\vec{y} \perp C(A)$, THEN $P\vec{y} = \vec{0}$

PROOF: SUPPOSE THAT $\vec{y} \in \mathbb{R}^m$ AND $\vec{y} \perp C(A)$
BY PROBLEM 3, $\vec{y} \perp C(A) \Rightarrow A^T\vec{y} = \vec{0}$.

$$\text{THUS } P\vec{y} = (A\,(A^TA)^{-1}A^T)\vec{y}$$
$$= A(A^TA)^{-1}(A^T\vec{y})$$
$$= A(A^TA)^{-1}\vec{0} = \vec{0}. \quad \blacksquare$$

(c) CLAIM: $\forall\, \vec{y} \in C(A)$, $P\vec{y} = \vec{y}$.

PROOF: LET $\vec{y} \in C(A)$ BE GIVEN; THEN $\exists\, \vec{x} \in \mathbb{R}^n$ WITH $A\vec{x} = \vec{y}$.

$$\text{THUS } P\vec{y} = A(A^TA)^{-1}A^T(A\vec{x})$$
$$= A\,\cancelto{I}{(A^TA)^{-1}(A^TA)}\,\vec{x}$$
$$= A\vec{x} = \vec{y}. \quad \blacksquare$$

(d) CLAIM: IF $\vec{p} \in C(A)$ AND $\vec{n} \perp C(A)$, THEN $P(\vec{p}+\vec{n}) = \vec{p}$

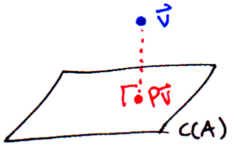
PROOF: SUPPOSE THAT $\vec{p} \in C(A)$ AND $\vec{n} \perp C(A)$.

THEN $P(\vec{p}+\vec{n}) = P\vec{p} + P\vec{n}$ BECAUSE $P$ IS A L.T.
$= \vec{p} + \vec{0}$ BY PARTS (c) + (b), RESPECTIVELY
$= \vec{p}. \quad \blacksquare$

NOW, FOR ANY $\vec{v} \in \mathbb{R}^m$, $\vec{v} = \text{PROJ}_{C(A)}\vec{v} + \text{ORTH}_{C(A)}\vec{v}$,
SO THE ABOVE RESULT TELLS US THAT

$$P\vec{v} = P(\underbrace{\text{PROJ}_{C(A)}\vec{v}}_{\in C(A)} + \underbrace{\text{ORTH}_{C(A)}\vec{v}}_{\perp C(A)}) = \text{PROJ}_{C(A)}\vec{v},$$

I.E., THE MATRIX $P$ GIVES US
PROJECTION ONTO $C(A)$!



7. • IF THE COLUMNS OF $A$ ARE ORTHONORMAL, THEN $A^TA = I_n$,
SO $P = A(A^TA)^{-1}A^T = A\,(I_n^{-1})\,A^T = AA^T$

THIS IS ACTUALLY OUR OLD PROJECTION FORMULA!
IF $A = [\vec{a}_1 | \dots | \vec{a}_n]$, THEN

$$AA^T\vec{x} = A\begin{bmatrix} \vec{a}_1\cdot\vec{x} \\ \vec{a}_2\cdot\vec{x} \\ \vdots \\ \vec{a}_n\cdot\vec{x}\end{bmatrix} = (\vec{a}_1\cdot\vec{x})\vec{a}_1 + (\vec{a}_2\cdot\vec{x})\vec{a}_2 + \dots + (\vec{a}_n\cdot\vec{x})\vec{a}_n$$

• IF THE COLUMNS OF $A$ FORM A BASIS FOR $\mathbb{R}^m$, THEN $A$ IS SQUARE AND INVERTIBLE; THIS MEANS THAT $A^T$ IS ALSO INVERTIBLE, FOR

$$AA^{-1} = I_m = A^{-1}A \xrightarrow{\text{TRANSPOSE}} (AA^{-1})^T = I_m^T = (A^{-1}A)^T$$
$$\Rightarrow (A^{-1})^T A^T = I_m = A^T (A^{-1})^T$$
$$\text{I.E., } (A^T)^{-1} = (A^{-1})^T.$$

NET EFFECT: $P = A\,(A^TA)^{-1}A^T$
$= A\,(A^{-1}(A^T)^{-1})\,A^T$
$= (\cancelto{I}{A\,A^{-1}})(\cancelto{I}{(A^T)^{-1}A^T})$
$= I$, WHICH ONLY MAKES SENSE, BECAUSE PROJECTING ONTO $C(A)$ IS JUST PROJECTING ONTO $\mathbb{R}^m$ — EVERY VECTOR IN $\mathbb{R}^m$ SHOULD PROJECT TO ITSELF!

8. GIVEN THE LINEAR SYSTEM $[\vec{A}\,|\,\vec{b}]$, EVEN IF IT'S INCONSISTENT, WE CAN SOLVE THE SYSTEM $A\vec{x} = P\vec{b}$, BECAUSE $P\vec{b} = \text{PROJ}_{C(A)}\vec{b} \in C(A)$.

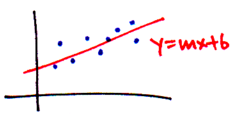
I.E., WE CAN FIND AN $\vec{x}$ THAT MAPS TO $P\vec{b}$, THE CLOSEST VECTOR IN $C(A)$ TO $\vec{b}$:

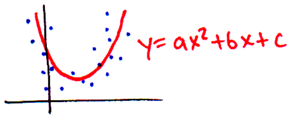
$$A\underbrace{\vec{x}}_{} = P\vec{b} = \underbrace{A(A^TA)^{-1}A^T\vec{b}}_{}, \quad \text{I.E., } \vec{x} = (A^TA)^{-1}A^T\vec{b},$$

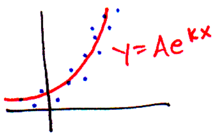
$$\text{I.E., } (A^TA)\vec{x} = A^T\vec{b},$$

BY SOLVING THE CONSISTENT SYSTEM $[A^TA\,|\,A^T\vec{b}]$.

*THIS ALLOWS US TO FIND THE "BEST FIT" OF A LINE OR CURVE TO SOME DATA POINTS BY SETTING UP A LINEAR SYSTEM AND USING PROJECTION TO FIND ITS BEST APPROXIMATE SOLUTION.







9. FITTING $y = \underset{\uparrow}{a}x + \underset{\uparrow}{b}$ TO $\{(1,1),(5,2),(7,3),(8,4),(15,10)\}$ GIVES

SOLVING FOR a, b

(x,y)

$$\begin{array}{ll} (1,1) & a+b=1 \\ (5,2) & 5a+b=2 \\ (7,3) & 7a+b=3 \\ (8,4) & 8a+b=4 \\ (15,10) & 15a+b=10 \end{array} \rightsquigarrow \left[\underbrace{\begin{array}{cc} a & b \\ 1 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \\ 15 & 1 \end{array}}_{A} \middle| \underbrace{\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 10 \end{array}}_{\vec{b}}\right]$$

TO FIT, SOLVE $[A^TA\,|\,A^T\vec{b}]$:

$$A^TA = \begin{bmatrix} 1 & 5 & 7 & 8 & 15 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & 1 \\ 7 & 1 \\ 8 & 1 \\ 15 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} 364 & 36 \\ 36 & 5 \end{bmatrix}$$

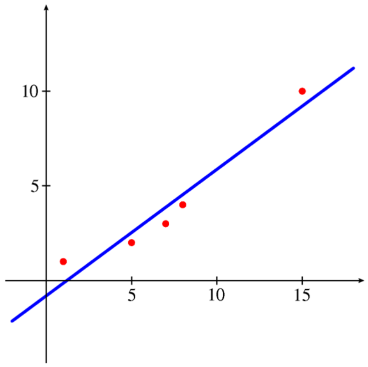
$$A^T\vec{b} = \begin{bmatrix} 1 & 5 & 7 & 8 & 15 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 10 \end{bmatrix} = \cdots = \begin{bmatrix} 214 \\ 20 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} a & b & \\ 364 & 36 & 214 \\ 36 & 5 & 20 \end{array}\right] \rightsquigarrow \begin{array}{l} a = \frac{175}{262} \\ b = -\frac{106}{131} \end{array}$$

∴ THE BEST FIT IS

$$y = \frac{175}{262}x - \frac{106}{131}$$

$$\approx 0.668x - 0.809$$



10. FITTING $\underset{\substack{\uparrow \\ \text{SOLVING FOR } a,b,c}}{y = ax^2 + bx + c}$ TO $\underset{(x,y)}{\{(0,0),(0,1),(3,3),(5,3),(6,-1)\}}$ GIVES

| | |
|---|---|
| $(0,0)$ | $0a + 0b + c = 0$ |
| $(0,1)$ | $0a + 0b + c = 1$ |
| $(3,3)$ | $9a + 3b + c = 3$ |
| $(5,3)$ | $25a + 5b + c = 3$ |
| $(6,-1)$ | $36a + 6b + c = -1$ |

$$\rightsquigarrow \begin{array}{ccc|c} a & b & c & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 9 & 3 & 1 & 3 \\ 25 & 5 & 1 & 3 \\ 36 & 6 & 1 & -1 \end{array} \qquad \underbrace{\phantom{xxx}}_{A} \ \underbrace{\phantom{x}}_{\vec{b}}$$

TO FIT, SOLVE $[A^TA \mid A^T\vec{b}]$:

$$A^TA = \begin{bmatrix} 0 & 0 & 9 & 25 & 36 \\ 0 & 0 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \\ 36 & 6 & 1 \end{bmatrix} = \dots = \begin{bmatrix} 2002 & 368 & 70 \\ 368 & 70 & 14 \\ 70 & 14 & 5 \end{bmatrix}$$

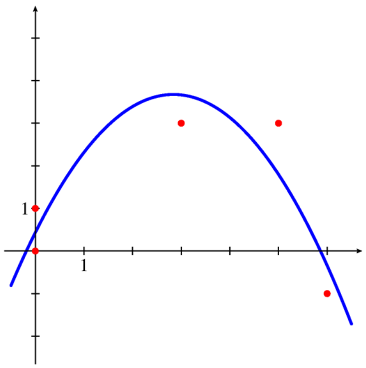
$$A^T\vec{b} = \begin{bmatrix} 0 & 0 & 9 & 25 & 36 \\ 0 & 0 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \dots = \begin{bmatrix} 66 \\ 18 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} a & b & c & \\ 2002 & 368 & 70 & 66 \\ 368 & 70 & 14 & 18 \\ 70 & 14 & 5 & 6 \end{array}\right] \rightsquigarrow \begin{array}{l} a = -\dfrac{317}{789} \\ b = \dfrac{1801}{789} \\ c = \dfrac{114}{263} \end{array}$$

$\therefore$ THE BEST FIT IS

$$y = -\frac{317}{789}x^2 + \frac{1801}{789}x + \frac{114}{263}$$

$$\approx -0.402x^2 + 2.283x + 0.433$$



11. FITTING $z = ax + by + c$ TO $\underset{(x,y,z)}{\{(0,0,0),(1,2,2),(3,2,1),(4,2,1)\}}$ GIVES

| | |
|---|---|
| $(0,0,0)$ | $0a + 0b + c = 0$ |
| $(1,2,2)$ | $1a + 2b + c = 2$ |
| $(3,2,1)$ | $3a + 2b + c = 1$ |
| $(4,2,1)$ | $4a + 2b + c = 1$ |

$$\rightsquigarrow \begin{array}{ccc|c} a & b & c & \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 1 \\ 4 & 2 & 1 & 1 \end{array} \qquad \underbrace{\phantom{xxx}}_{A} \ \underbrace{\phantom{x}}_{\vec{b}}$$

TO FIT, SOLVE $[A^TA \mid A^T\vec{b}]$:

$$A^TA = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \dots = \begin{bmatrix} 26 & 16 & 8 \\ 16 & 12 & 6 \\ 8 & 6 & 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \dots = \begin{bmatrix} 9 \\ 8 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} a & b & c & \\ 26 & 16 & 8 & 9 \\ 16 & 12 & 6 & 8 \\ 8 & 6 & 4 & 4 \end{array}\right] \rightsquigarrow \begin{array}{l} a = -\dfrac{5}{14} \\ b = \dfrac{8}{7} \\ c = 0 \end{array}$$

$\therefore$ THE BEST FIT IS $\underline{z = -\frac{5}{14}x + \frac{8}{7}y}$

12. FITTING $y = Ae^{kx}$ TO $\{\underset{(x,y)}{(0,1)}, (2,e), (4,e^{3/2}), (5,e^2)\}$ GIVES

$$\Updownarrow$$

$$\ln y = \underbrace{\ln A}_{\uparrow} + \underset{\uparrow}{k}x$$

SOLVING FOR
(ln A) + K

$(0,1)$ $(\ln A) + 0k = 0$

$(2,e)$ $(\ln A) + 2k = 1$

$(4,e^{3/2})$ $(\ln A) + 4k = \frac{3}{2}$

$(5,e^2)$ $(\ln A) + 5k = 2$

$$\rightsquigarrow \begin{array}{cc|c} (\ln A) & k & \\ 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 3/2 \\ 1 & 5 & 2 \end{array} \qquad \underbrace{\quad}_{A} \; \underbrace{\;}_{\vec{b}}$$

TO FIT, SOLVE $[A^TA \mid A^T\vec{b}]$:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} = \cdots = \begin{bmatrix} 4 & 11 \\ 11 & 45 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 3/2 \\ 2 \end{bmatrix} = \cdots = \begin{bmatrix} 9/2 \\ 18 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} (\ln A) & k & \\ 4 & 11 & 9/2 \\ 11 & 45 & 18 \end{array}\right] \rightsquigarrow \begin{array}{l} \ln A = \frac{9}{118} \Rightarrow A = e^{9/118} \\ k = \frac{45}{118} \end{array}$$

$\therefore$ THE BEST FIT IS

$$y = e^{9/118} e^{\frac{45}{118}x}$$

$$\approx 1.070 e^{0.381x}$$