1. [$\vec n \perp W$ MEANS $\forall \vec w \in W$, $\vec n \perp \vec w$.]

CLAIM: IF $\vec n \perp W$, THEN $\forall\, \vec w \in W$, $\|\vec n + \vec w\| \geq \|\vec n\|$

PROOF: SUPPOSE THAT $\vec n \perp W$, I.E., $\forall\, \vec w \in W$, $\vec n \perp \vec w$,
I.E., $\forall\, \vec w \in W$, $\langle \vec n, \vec w\rangle = 0$. $(*)$

LET $\vec w \in W$ BE GIVEN.

[GOAL: $\|\vec n + \vec w\| \geq \|\vec n\|$, I.E.,
$\|\vec n + \vec w\|^2 \geq \|\vec n\|^2$, I.E.,
$\langle \vec n + \vec w, \vec n + \vec w\rangle \geq \langle \vec n, \vec n\rangle$

THEN $\langle \vec n + \vec w, \vec n + \vec w\rangle$

$= \langle \vec n, \vec n + \vec w\rangle + \langle \vec w, \vec n + \vec w\rangle$ (BILINEARITY)

$= \langle \vec n, \vec n\rangle + \langle \vec n, \vec w\rangle + \langle \vec w, \vec n\rangle + \langle \vec w, \vec w\rangle$ (SYMMETRY)

$= \langle \vec n, \vec n\rangle + 2\cancelto{0 \text{ BY } (*)}{\langle \vec n, \vec w\rangle} + \langle \vec w, \vec w\rangle$

$= \langle \vec n, \vec n\rangle + \langle \vec w, \vec w\rangle$

$\geq \langle \vec n, \vec n\rangle$ BECAUSE $\langle \vec w, \vec w\rangle \geq 0$, (POSITIVE-DEFINITE)

SO $\|\vec n + \vec w\| \geq \|\vec n\|$. ■

2. [$\mathcal{B} = (\vec w_1, \vec w_2, \ldots, \vec w_n)$: ORTHONORMAL BASIS FOR $W < V$.]

(a) $\mathrm{PROJ}_W \vec v \overset{\text{DEF}}{=} \underbrace{\langle \vec v, \vec w_1\rangle \vec w_1}_{\mathrm{PROJ}_{\vec w_1} \vec v} + \underbrace{\langle \vec v, \vec w_2\rangle \vec w_2}_{\mathrm{PROJ}_{\vec w_2} \vec v} + \cdots + \underbrace{\langle \vec v, \vec w_n\rangle \vec w_n}_{\mathrm{PROJ}_{\vec w_n} \vec v}$.

(b) $\mathrm{ORTH}_W \vec v = \vec v - \mathrm{PROJ}_W \vec v = \vec v - \langle \vec v, \vec w_1\rangle \vec w_1 - \cdots - \langle \vec v, \vec w_n\rangle \vec w_n$

P.S. 23, #9 SHOWS THAT $\forall\, \vec w \in W$, $\mathrm{ORTH}_W \vec v \perp \vec w$,
I.E., THAT $\mathrm{ORTH}_W \vec v \perp W$.

(c) CLAIM: $\forall\, \vec w \in W$, $\|\vec v - \vec w\| \geq \|\vec v - \mathrm{PROJ}_W \vec v\|$

PROOF: LET $\vec w \in W$ BE GIVEN. [GOAL: $\|\vec v - \vec w\| \geq \|\vec v - \mathrm{PROJ}_W \vec v\|$

THEN $\|\vec v - \vec w\|$

$= \|(\mathrm{PROJ}_W \vec v + \mathrm{ORTH}_W \vec v) - \vec w\|$ $(\vec v = \mathrm{PROJ}_W \vec v + \mathrm{ORTH}_W \vec v)$

$= \|\mathrm{ORTH}_W \vec v + (\mathrm{PROJ}_W \vec v - \vec w)\|$ (REARRANGE)

BUT $\mathrm{ORTH}_W \vec v \perp W$ FROM PART (b)
AND $\mathrm{PROJ}_W \vec v - \vec w \in W$, SO PROBLEM 1 $\begin{pmatrix} \vec n \leftarrow \mathrm{ORTH}_W \vec v \\ \vec w \leftarrow \mathrm{PROJ}_W \vec v - \vec w \end{pmatrix}$
TELLS US THAT THIS IS

$\geq \|\mathrm{ORTH}_W \vec v\| = \|\vec v - \mathrm{PROJ}_W \vec v\|$ BY DEFINITION. ■

$*$ THIS TELLS US THAT $\mathrm{PROJ}_W \vec v$ IS THE VECTOR OF W THAT IS CLOSEST TO (I.E., BEST APPROXIMATES) THE VECTOR $\vec v \in V$.

3. $\mathcal{F}_n = \{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx\} \subset C([-\pi,\pi])$;

$\langle f,g\rangle \overset{DEF}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} fg$; $\quad f(x) \in C([-\pi,\pi])$,

n^{TH}-ORDER FOURIER POLYNOMIAL FOR $f(x)$: $F_n(x) \overset{DEF}{=} \mathrm{PROJ}_{\mathrm{SPAN}\,\mathcal{F}_n} f(x)$.

(a) $F_n(x) = \mathrm{PROJ}_{\mathrm{SPAN}\,\mathcal{F}_n} f(x)$

$$= \langle f(x), \tfrac{1}{\sqrt{2}}\rangle \tfrac{1}{\sqrt{2}} + \langle f(x), \cos x\rangle \cos x + \langle f(x), \sin x\rangle \sin x + \dots + \langle f(x), \cos nx\rangle \cos nx + \langle f(x), \sin nx\rangle \sin nx.$$

(BY THE USUAL PROJECTION FORMULA IN 2(a))

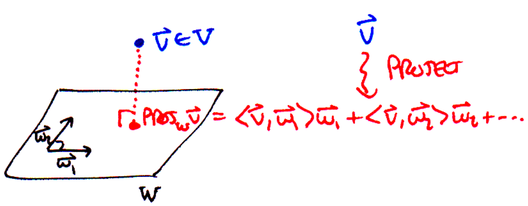
(b) $F_n(x) \in \mathrm{SPAN}\,\mathcal{F}_n$ IS A LINEAR COMBINATION OF THE FUNCTIONS

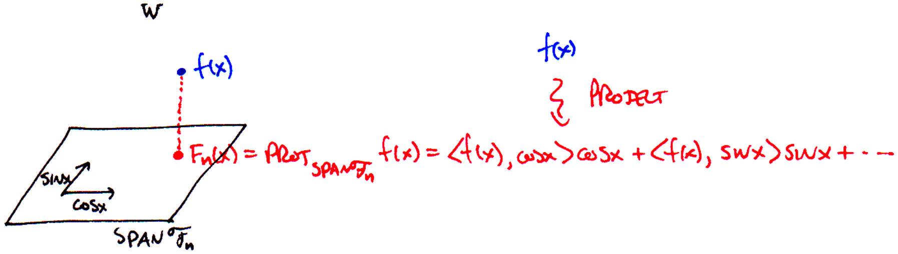
$\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx$;

IT IS, BY 2(c), THE LINEAR COMBINATION THAT BEST APPROXIMATES $f(x)$.

(c) AS n INCREASES, THE COLLECTION $\mathcal{F}_n$ GROWS, AND THUS SO DOES THE DIMENSION ("SIZE") OF ITS SPAN... THUS, AS n INCREASES, SPAN $\mathcal{F}_n$ WILL FILL UP MORE AND MORE OF $C([-\pi,\pi])$, SO WE EXPECT THAT $F_n(x)$ WILL BETTER AND BETTER APPROXIMATE $f(x)$.

THIS IS ALL JUST AS IN ANY OTHER INNER PRODUCT SPACE:





$f(x) = x \in C([-\pi,\pi])$, $\quad \langle f,g\rangle \overset{DEF}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} fg$

4. $\langle x, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} x\,dx = \underline{0}$

5. $\langle x, \cos kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx = \underline{0}$

ODD FUNCTIONS INTEGRATED OVER THE INTERVAL $[-\pi,\pi]$

6. $\langle x, \sin kx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx\,dx$

$$= \frac{1}{\pi}\left[\frac{1}{k^2}\sin kx - \frac{1}{k}x\cos kx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(\cancelto{0}{\frac{1}{k^2}\sin k\pi} - \frac{\pi}{k}\cos k\pi\right) - \left(\cancelto{0}{\frac{1}{k^2}\sin(-k\pi)} + \frac{\pi}{k}\underbrace{\cos(-k\pi)}_{=\cos(k\pi)}\right)\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{k}\cos k\pi\right] = -\frac{2}{k}\cos k\pi \qquad (\cos k\pi = (-1)^k)$$

$$= -\frac{2}{k}(-1)^k = (-1)^{k+1}\cdot\frac{2}{k}$$

7. $F_4(x) = \cancelto{0}{\langle x, \tfrac{1}{\sqrt{2}}\rangle}\frac{1}{\sqrt{2}} + \cancelto{0}{\langle x, \cos x\rangle}\cos x + \langle x, \sin x\rangle \sin x$
$+ \cancelto{0}{\langle x, \cos 2x\rangle}\cos 2x + \langle x, \sin 2x\rangle \sin 2x$
$+ \cancelto{0}{\langle x, \cos 3x\rangle}\cos 3x + \langle x, \sin 3x\rangle \sin 3x$
$+ \cancelto{0}{\langle x, \cos 4x\rangle}\cos 4x + \langle x, \sin 4x\rangle \sin 4x$

$$= \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x$$

8. (a) AS IN 7, THE $\frac{1}{\sqrt{2}}$ AND ALL COSINE TERMS DROP OUT, LEAVING:

$$F_n(x) = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \dots + (-1)^{n+1}\frac{2}{n}\sin nx$$

(b) VIA THE GENERALIZED PYTHAGOREAN THEOREM, FOR ANY ORTHOGONAL COLLECTION $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$,

$$\|\vec{w}_1 + \vec{w}_2 + \dots + \vec{w}_n\|^2 = \|\vec{w}_1\|^2 + \|\vec{w}_2\|^2 + \dots + \|\vec{w}_n\|^2, \text{ SO} \dots$$

$$\|F_n(x)\|^2 = \left\|\frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \dots + (-1)^{n+1}\frac{2}{n}\sin nx\right\|^2$$

$$= \left\|\frac{2}{1}\sin x\right\|^2 + \left\|\frac{2}{2}\sin 2x\right\|^2 + \left\|\frac{2}{3}\sin 3x\right\|^2 + \dots + \left\|(-1)^{n+1}\frac{2}{n}\sin nx\right\|^2$$

$$= \frac{4}{1^2}\|\sin x\|^2 + \frac{4}{2^2}\|\sin 2x\|^2 + \frac{4}{3^2}\|\sin 3x\|^2 + \dots + \frac{4}{n^2}\|\sin nx\|^2$$

$$= \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \dots + \frac{4}{n^2}$$

SIN KX IS A UNIT VECTOR, SO $\|\sin kx\| = 1$

$$= 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right]$$

(c) $$\lim_{n\to\infty}\|F_n(x)\|^2 = \lim_{n\to\infty} 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right]$$

$$= 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

(d) COMPUTING $\|f(x)\|^2$ DIRECTLY,

$$\|f(x)\|^2 = \langle f(x), f(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{1}{3}\pi^3\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{3}\pi^3 + \frac{1}{3}\pi^3\right] = \frac{2}{3}\pi^2$$

(e) (BECAUSE $F_n(x) \overset{\text{SHOULD}}{\rightsquigarrow} f(x)$, $\|F_n(x)\|^2 \overset{\text{SHOULD}}{\rightsquigarrow} \|f(x)\|$:)

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\pi^2 \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad (!)$$