V: INNER PRODUCT SPACE

1. TWO VECTORS $\vec{v}, \vec{w} \in V$ ARE ORTHOGONAL IF $\langle \vec{v}, \vec{w} \rangle = 0$; WE DENOTE THIS BY $\vec{v} \perp \vec{w}$.

2. (a) $\vec{w} \in V$ IS A UNIT VECTOR IF $\|\vec{w}\| = 1$, I.E., IF $\langle \vec{w}, \vec{w} \rangle = 1$.

(b) IF $\vec{v} \in V$ IS ANY NONZERO VECTOR, THEN $\frac{1}{\|\vec{v}\|}\vec{v}$ IS A UNIT VECTOR IN THE SAME DIRECTION; IT'S IN THE SAME DIRECTION BECAUSE IT'S A POSITIVE MULTIPLE OF $\vec{v}$ ($\|\vec{v}\| > 0$), AND IT'S A UNIT VECTOR BECAUSE:

$$\left\langle \frac{1}{\|\vec{v}\|}\vec{v}, \frac{1}{\|\vec{v}\|}\vec{v} \right\rangle = \frac{1}{\|\vec{v}\|^2}\langle \vec{v}, \vec{v} \rangle = \frac{\langle \vec{v}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} = 1 \checkmark.$$

(c) IF $\vec{w}$ IS A UNIT VECTOR, $\mathrm{PROJ}_{\vec{w}}\vec{v} \overset{\mathrm{DEF}}{=} \frac{\langle \vec{v}, \vec{w} \rangle}{\underbrace{\langle \vec{w}, \vec{w} \rangle}_{1}} \vec{w} = \langle \vec{v}, \vec{w} \rangle \vec{w}$.

3. A COLLECTION $\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ IS ORTHONORMAL IF:

$$\left.\begin{array}{l} \bullet\ \forall\, i \neq j,\ \langle \vec{v}_i, \vec{v}_j \rangle = 0 \\ \text{AND } \bullet\ \forall\, i,\ \langle \vec{v}_i, \vec{v}_i \rangle = 1 \end{array}\right\} \text{ I.E., } \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij} \overset{\mathrm{DEF}}{=} \begin{cases} 0 & \text{IF } i \neq j \\ 1 & \text{IF } i = j \end{cases}$$

4. THE GRAM-SCHMIDT ORTHOGONALIZATION PROCESS TAKES ANY COLLECTION $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ AND PRODUCES AN ORTHONORMAL COLLECTION $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ WITH THE SAME SPAN (I.E., AN ORTHONORMAL BASIS FOR $\mathrm{SPAN}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$). IT PROCEEDS AS FOLLOWS:

ITERATIVELY TAKE EACH VECTOR $\vec{v}_i$ OF THE ORIGINAL COLLECTION AND:

- SUBTRACT ITS PROJECTIONS ONTO ALL $\vec{u}_j$ SO FAR OBTAINED:

$$\vec{v}_i \rightsquigarrow \vec{v}_i' = \vec{v}_i - \langle \vec{v}_i, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_i, \vec{u}_2 \rangle \vec{u}_2 - \cdots$$

($\vec{v}_i'$ IS NOW $\perp$ TO ALL OF THE VECTORS $\vec{u}_j$)

- IF $\vec{v}_i' = \vec{0}$, TOSS IT OUT AND MOVE ON. OTHERWISE, DIVIDE $\vec{v}_i'$ BY ITS LENGTH AND ADD IT TO YOUR ORTHONORMAL COLLECTION:

$$\vec{u}_j = \frac{1}{\|\vec{v}_i'\|}\vec{v}_i'.$$

5. [$\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$: ORTHONORMAL BASIS FOR V]

(a) WHEN $\mathcal{B}$ IS AN ORTHONORMAL BASIS FOR V, THE ISOMORPHISMS $[\mathcal{B}]$ AND $[\mathcal{B}]^{-1}$ ARE ISOMETRIES — I.E., THEY RESPECT NOT ONLY LINEAR COMBINATIONS, BUT LENGTHS, ANGLES, ETC., ALLOWING US TO TRANSLATE EVEN PROBLEMS INVOLVING INNER PRODUCTS INTO PROBLEMS IN $\mathbb{R}^n$!

(b) $$[\mathcal{B}]^{-1}\vec{v} = \begin{bmatrix} \langle \vec{v}, \vec{v}_1 \rangle \\ \langle \vec{v}, \vec{v}_2 \rangle \\ \vdots \\ \langle \vec{v}, \vec{v}_n \rangle \end{bmatrix}.$$

WHY? WRITING $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$ AND TAKING THE INNER PRODUCT WITH $\vec{v}_i$ GIVES $\langle \vec{v}, \vec{v}_i \rangle = a_i = i^{\mathrm{TH}}$ ENTRY OF $[\mathcal{B}]^{-1}\vec{v}$.

(c) TO FIND AN ORTHONORMAL BASIS FOR ANY FINITE-DIMENSIONAL INNER PRODUCT SPACE V, JUST TAKE ANY BASIS (OR EVEN A SPANNING SET) FOR V AND RUN IT THROUGH GRAM-SCHMIDT!

6. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ BECAUSE $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = -1 + 0 + 1 = \underline{\underline{0}}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \not\perp \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ BECAUSE $\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\rangle = 2 - 1 + 1 = \underline{\underline{2}} \neq 0$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ HAS NORM $\sqrt{1+1+1} = \sqrt{3}$, SO $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ IS A UNIT VECTOR IN ITS DIRECTION.

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ HAS NORM $\sqrt{1+0+1} = \sqrt{2}$, SO $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ IS A UNIT VECTOR IN ITS DIRECTION.

$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ HAS NORM $\sqrt{4+1+1} = \sqrt{6}$, SO $\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ IS A UNIT VECTOR IN ITS DIRECTION.

7. $$\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 \Leftrightarrow \langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle = \langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle$$

$$\Leftrightarrow \cancel{\langle \vec{v}, \vec{v} \rangle} + 2\langle \vec{v}, \vec{w} \rangle + \cancel{\langle \vec{w}, \vec{w} \rangle} = \cancel{\langle \vec{v}, \vec{v} \rangle} + \cancel{\langle \vec{w}, \vec{w} \rangle}$$

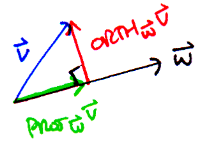
$$\Leftrightarrow 2\langle \vec{v}, \vec{w} \rangle = 0 \Leftrightarrow \langle \vec{v}, \vec{w} \rangle = 0 \Leftrightarrow \vec{v} \perp \vec{w}. \blacksquare$$

8. SUPPOSE THAT $\vec v, \vec w \in V$ WITH $\vec w \neq \vec 0$, AND LET $\text{ORTH}_{\vec w}\vec v = \vec v - \text{PROJ}_{\vec w}\vec v$.

THEN $\langle \text{ORTH}_{\vec w}\vec v, \vec w\rangle = \langle \vec v - \text{PROJ}_{\vec w}\vec v, \vec w\rangle$

$$= \langle \vec v,\vec w\rangle - \langle \text{PROJ}_{\vec w}\vec v,\vec w\rangle = \langle\vec v,\vec w\rangle - \left\langle \frac{\langle \vec v,\vec w\rangle}{\langle\vec w,\vec w\rangle}\vec w,\vec w\right\rangle$$

$$= \langle\vec v,\vec w\rangle - \frac{\langle\vec v,\vec w\rangle}{\cancel{\langle\vec w,\vec w\rangle}}\cancel{\langle\vec w,\vec w\rangle} = \langle\vec v,\vec w\rangle - \langle\vec v,\vec w\rangle = 0;$$



IN SUMMARY, $\langle \text{ORTH}_{\vec w}\vec v,\vec w\rangle = 0$, SO $\text{ORTH}_{\vec w}\vec v \perp \vec w$. ■

9. LET $\vec v \in V$,

AND LET $\mathcal{C} = \{\vec v_1, \vec v_2, \dots, \vec v_k\}$ BE AN ORTHONORMAL COLLECTION IN V.

SET $\vec v\,' = \vec v - \langle\vec v,\vec v_1\rangle\vec v_1 - \langle\vec v,\vec v_2\rangle\vec v_2 - \dots - \langle\vec v,\vec v_k\rangle\vec v_k$.

(a) CLAIM: $\forall i$, $\vec v\,' \perp \vec v_i$ (I.E., $\langle\vec v\,',\vec v_i\rangle = 0$)

PROOF: LET i BE GIVEN. THEN

$$\langle \vec v\,',\vec v_i\rangle = \langle \vec v - \langle\vec v,\vec v_1\rangle\vec v_1 - \langle\vec v,\vec v_2\rangle\vec v_2 - \dots - \langle\vec v,\vec v_k\rangle\vec v_k, \vec v_i\rangle$$

$$= \langle\vec v,\vec v_i\rangle - \langle\vec v,\vec v_1\rangle\langle\vec v_1,\vec v_i\rangle - \langle\vec v,\vec v_2\rangle\langle\vec v_2,\vec v_i\rangle - \dots - \langle\vec v,\vec v_k\rangle\langle\vec v_k,\vec v_i\rangle$$

$$= \langle\vec v,\vec v_i\rangle - \langle\vec v,\vec v_i\rangle\overset{1}{\cancel{\langle\vec v_i,\vec v_i\rangle}}$$ (ALL OTHERS $=0$ BECAUSE $\langle\vec v_i,\vec v_j\rangle=0$ FOR $i\neq j$)

$$= \langle\vec v,\vec v_i\rangle - \langle\vec v,\vec v_i\rangle = 0,$$

SO BY DEFINITION, $\vec v\,'\perp\vec v_i$. ■

(b) CLAIM: $\forall\, \vec w \in \text{SPAN}\,\mathcal C$, $\vec v\,'\perp\vec w$.

PROOF: LET $\vec w\in \text{SPAN}\,\mathcal C$ BE GIVEN. [GOAL: $\vec v\,'\perp\vec w$, I.E., $\langle\vec v\,',\vec w\rangle=0$]

THEN $\exists$ SCALARS $a_1,\dots,a_k$ WITH $\vec w = a_1\vec v_1 + a_2\vec v_2 + \dots + a_k\vec v_k$.

WELL, $\langle\vec v\,',\vec w\rangle = \langle\vec v\,', a_1\vec v_1 + a_2\vec v_2 + \dots + a_k\vec v_k\rangle$

$$= a_1\langle\vec v\,',\vec v_1\rangle + a_2\langle\vec v\,',\vec v_2\rangle + \dots + a_k\langle\vec v\,',\vec v_k\rangle$$

$$= a_1\cdot 0 + a_2\cdot 0 + \dots + a_k\cdot 0$$ (PART (a))

$$= 0,$$

SO BY DEFINITION, $\vec v\,'\perp\vec w$. ■

10. CLAIM: IF $\mathcal C = \{\vec v_1,\vec v_2,\dots,\vec v_n\}$ IS AN ORTHONORMAL COLLECTION THEN $\mathcal C$ IS LINEARLY INDEPENDENT.

PROOF: SUPPOSE THAT $\mathcal C = \{\vec v_1,\vec v_2,\dots,\vec v_n\}$ IS AN ORTHONORMAL COLLECTION.

[GOAL: $\mathcal C$ IS LINEARLY INDEPENDENT, I.E., $a_1\vec v_1 + a_2\vec v_2 + \dots + a_n\vec v_n = \vec 0 \Rightarrow a_1, a_2, \dots, a_n = 0$.

SUPPOSE THAT $a_1\vec v_1 + a_2\vec v_2 + \dots + a_n\vec v_n = \vec 0$.

FOR ANY i, IF WE TAKE THE INNER PRODUCT OF BOTH SIDES WITH $\vec v_i$, $\langle a_1\vec v_1 + a_2\vec v_2 + \dots + a_n\vec v_n, \vec v_i\rangle = \langle\vec 0,\vec v_i\rangle = 0$,

SO $a_1\langle\vec v_1,\vec v_i\rangle + a_2\langle\vec v_2,\vec v_i\rangle + \dots + a_n\langle\vec v_n,\vec v_i\rangle = 0$

I.E., $a_i\langle\vec v_i,\vec v_i\rangle = 0$ BECAUSE FOR $j\neq i$, $\langle\vec v_j,\vec v_i\rangle = 0$

THUS $a_i = 0$ SINCE $\langle\vec v_i,\vec v_i\rangle = 1$.

IN CONCLUSION, $a_1, a_2, \dots, a_n = 0$,

SO BY DEFINITION, $\mathcal C$ IS LINEARLY INDEPENDENT.

11. $\mathcal B = \left(\frac{1}{\sqrt2}, \cos x, \sin x, \cos 2x, \sin 2x\right) \subset C([-\pi,\pi])$,

WITH INNER PRODUCT $\langle f,g\rangle \overset{\text{DEF}}{=} \frac1\pi\int_{-\pi}^{\pi} fg$.

(a) TO SHOW THAT $\mathcal B$ IS AN ORTHONORMAL COLLECTION, CHECK ALL INNER PRODUCTS:

$$\left\langle \tfrac1{\sqrt2},\tfrac1{\sqrt2}\right\rangle = \frac1\pi\int_{-\pi}^\pi \frac12\, dx = \frac1\pi\left[\frac12 x\right]_{-\pi}^\pi = \frac1\pi\left[\frac\pi2+\frac\pi2\right] = 1 \checkmark$$

$$\left\langle\tfrac1{\sqrt2},\cos x\right\rangle = \frac1\pi\int_{-\pi}^\pi \frac1{\sqrt2}\cos x\,dx = \frac1{\pi\sqrt2}\left[\sin x\right]_{-\pi}^\pi = \frac1{\pi\sqrt2}\{0-0\} = 0 \checkmark$$

$$\left\langle\tfrac1{\sqrt2},\sin x\right\rangle = \frac1\pi\int_{-\pi}^\pi \frac1{\sqrt2}\sin x\,dx = \frac1{\pi\sqrt2}\left[-\cos x\right]_{-\pi}^\pi = \frac1{\pi\sqrt2}[-1+1] = 0 \checkmark$$

$$\left\langle\tfrac1{\sqrt2},\cos 2x\right\rangle = \frac1\pi\int_{-\pi}^\pi \frac1{\sqrt2}\cos 2x\,dx = \frac1{\pi\sqrt2}\left\{\frac12\sin 2x\right\}_{-\pi}^\pi = \frac1{\pi\sqrt2}[0-0] = 0 \checkmark$$

$$\left\langle\tfrac1{\sqrt2},\sin 2x\right\rangle = \frac1\pi\int_{-\pi}^\pi \frac1{\sqrt2}\sin 2x\,dx = \frac1{\pi\sqrt2}\left[-\frac12\cos 2x\right]_{-\pi}^\pi = \frac1{\pi\sqrt2}\left[-\frac12+\frac12\right] = 0 \checkmark$$

$$\langle \cos x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{\pi}\left[\frac{x}{2}+\frac{1}{4}\sin 2x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1 \checkmark$$

$$\langle \sin x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 x\,dx = \frac{1}{\pi}\left[\frac{x}{2}-\frac{1}{4}\sin 2x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1 \checkmark$$

$$\langle \cos 2x, \cos 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 2x\,dx = \frac{1}{\pi}\left[\frac{x}{2}+\frac{1}{8}\sin 4x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1 \checkmark$$

$$\langle \sin 2x, \sin 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2 2x\,dx = \frac{1}{\pi}\left[\frac{x}{2}-\frac{1}{8}\sin 4x\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1 \checkmark$$

$$\langle \sin x, \cos x\rangle = \langle \cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos x\sin x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\sin 2x\,dx$$
$$= \frac{1}{2\pi}\left[-\frac{1}{2}\cos 2x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[-\frac{1}{2}+\frac{1}{2}\right] = 0 \checkmark$$

$$\langle \sin 2x, \cos 2x\rangle = \langle \cos 2x, \sin 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos 2x\sin 2x\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}\sin 4x\,dx = \frac{1}{2\pi}\left[-\frac{1}{4}\cos 4x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[-\frac{1}{4}+\frac{1}{4}\right] = 0 \checkmark$$

$$\langle \sin x, \cos 2x\rangle = \langle \cos 2x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos 2x\sin x\,dx$$
$$= \frac{1}{\pi}\left[\frac{1}{2}\sin x - \frac{1}{6}\sin 3x\right]_{-\pi}^{\pi} = \frac{1}{\pi}[0-0] = 0 \checkmark$$

$$\langle \sin 2x, \cos x\rangle = \langle \cos x, \sin 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos x\sin 2x\,dx$$
$$= \frac{1}{\pi}\left[-\frac{1}{2}\sin x - \frac{1}{6}\sin 3x\right]_{-\pi}^{\pi} = \frac{1}{\pi}[0-0] = 0 \checkmark$$

(6) (SEE PROBLEM 5)

IF $f(x) = a_0\cdot\frac{1}{\sqrt{2}} + a_1\cos x + a_2\sin x + a_3\cos 2x + a_4\sin 2x$,

THEN $a_0 = \langle f(x), \frac{1}{\sqrt{2}}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}f(x)\,dx$,

$a_1 = \langle f(x), \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$,

$a_2 = \langle f(x), \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$,

$a_3 = \langle f(x), \cos 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx$, AND

$a_4 = \langle f(x), \sin 2x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx$.

12. TAKE THE COLLECTION $\{1, x, x^2, x^3\}$ AND APPLY GRAM-SCHMIDT:

[NOTE THAT $\int_0^1 x^a\,dx = \frac{1}{a+1}$]

$\underline{1}$: $\langle 1, 1\rangle = \int_0^1 1\,dx = 1$, SO $\|1\| = 1$,

I.E., THIS IS ALREADY A UNIT VECTOR: LET $\vec{u}_1 = 1$.

$\underline{x}$: • SUBTRACT PROJECTION ONTO $\vec{u}_1$:

$$x \rightsquigarrow x - \langle x, 1\rangle 1 = x - \left(\int_0^1 x\,dx\right)1 = x - \frac{1}{2}.$$

• NORMALIZE: $\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle = \int_0^1 (x-\frac{1}{2})^2\,dx = \frac{1}{3}(x-\frac{1}{2})^3\Big|_0^1$

$$= \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right] = \frac{1}{3}\left[\frac{1}{8}+\frac{1}{8}\right] = \frac{1}{12},$$

SO $\|x-\frac{1}{2}\| = \frac{1}{\sqrt{12}} = \frac{1}{\sqrt{4\cdot 3}} = \frac{1}{2\sqrt{3}}$.

THUS WE SET $\vec{u}_2 = \frac{1}{\frac{1}{2\sqrt{3}}}(x-\frac{1}{2}) = 2\sqrt{3}(x-\frac{1}{2}) = \sqrt{3}(2x-1)$.

$\underline{x^2}$: • SUBTRACT PROJECTIONS ONTO $\vec{u}_1$ AND $\vec{u}_2$:

$$x^2 \rightsquigarrow x^2 - \langle x^2, 1\rangle 1 - \langle x^2, \sqrt{3}(2x-1)\rangle\sqrt{3}(2x-1)$$
$$= x^2 - \langle x^2, 1\rangle - 3\langle x^2, 2x-1\rangle(2x-1)$$
$$= x^2 - \left(\int_0^1 x^2\,dx\right) - 3\left(\int_0^1 2x^3 - x^2\,dx\right)(2x-1)$$
$$= x^2 - \frac{1}{3} - 3\left(\frac{1}{2}-\frac{1}{3}\right)(2x-1)$$
$$= x^2 - \frac{1}{3} - \cancel{3\left(\frac{1}{6}\right)}^{\frac{1}{2}}(2x-1) = x^2 - x + \frac{1}{6}.$$

• NORMALIZE: $\langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\rangle = \int_0^1 (x^2-x+\frac{1}{6})^2\,dx$

$$= \int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\,dx$$
$$= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{1}{180},$$

SO $\|x^2-x+\frac{1}{6}\| = \frac{1}{\sqrt{180}} = \frac{1}{\sqrt{36\cdot 5}} = \frac{1}{6\sqrt{5}}$

THUS WE SET $\vec{u}_3 = \frac{1}{1/6\sqrt{5}}(x^2-x+\frac{1}{6}) = 6\sqrt{5}(x^2-x+\frac{1}{6})$

$$= \sqrt{5}(6x^2-6x+1)$$

$\underline{x^3}$: • $\underline{\text{SUBTRACT PROJECTIONS ONTO } \vec{u}_1, \vec{u}_2, \vec{u}_3}$:

$$x^3 \rightsquigarrow x^3 - \langle x^3, 1\rangle 1 - \langle x^3, \sqrt{3}(2x-1)\rangle \sqrt{3}\langle 2x-1\rangle$$
$$- \langle x^3, \sqrt{5}(6x^2-6x+1)\rangle \sqrt{5}(6x^2-6x+1)$$
$$= x^3 - \langle x^3, 1\rangle - 3\langle x^3, 2x-1\rangle(2x-1) - 5\langle x^3, 6x^2-6x+1\rangle(6x^2-6x+1)$$
$$= x^3 - \int_0^1 x^3 dx - 3\left(\int_0^1 2x^4 - x^3 dx\right)(2x-1) - 5\left(\int_0^1 6x^5 - 6x^4 + x^3 dx\right)(6x^2-6x+1)$$
$$= x^3 - \frac{1}{4} - 3\left(\frac{2}{5} - \frac{1}{4}\right)(2x-1) - 5\left(1 - \frac{6}{5} + \frac{1}{4}\right)(6x^2-6x+1)$$
$$= x^3 - \frac{1}{4} - \frac{9}{20}(2x-1) - \frac{1}{4}(6x^2-6x+1)$$
$$= x^3 - \frac{1}{4} - \frac{9}{10}x + \frac{9}{20} - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{1}{4}$$
$$= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

• $\underline{\text{NORMALIZE}}$:

$$\left\langle x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}, x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right\rangle = \int_0^1 \left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)^2 dx$$
$$= \int_0^1 x^6 - 3x^5 + \frac{69}{20}x^4 - \frac{19}{10}x^3 + \frac{51}{100}x^2 - \frac{3}{50}x + \frac{1}{400} dx$$
$$= \frac{1}{7} - \frac{1}{2} + \frac{69}{100} - \frac{19}{40} + \frac{17}{100} - \frac{3}{100} + \frac{1}{400} = \frac{1}{2800}$$
$$\therefore \left\|x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right\| = \frac{1}{\sqrt{2800}} = \frac{1}{\sqrt{400\cdot 7}} = \frac{1}{20\sqrt{7}}$$

SO WE SET
$$\underset{\sim}{\vec{u}_4} = \frac{1}{1/20\sqrt{7}}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)$$
$$= 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right)$$
$$= \underset{\sim}{\sqrt{7}(20x^3 - 30x^2 + 12x - 1)}$$

13. APPLY GRAM-SCHMIDT TO $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$:

$\underline{\begin{bmatrix}-1\\1\\0\end{bmatrix}}$: $\left\|\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\| = \sqrt{1+1+0} = \sqrt{2}$. THUS, SET $\underset{\sim}{\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}}$.

$\underline{\begin{bmatrix}0\\-1\\1\end{bmatrix}}$: • $\underline{\text{SUBTRACT PROJECTION ONTO } \vec{u}_1}$:

$$\begin{bmatrix}0\\-1\\1\end{bmatrix} \rightsquigarrow \begin{bmatrix}0\\-1\\1\end{bmatrix} - \left\langle \begin{bmatrix}0\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\rangle \left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$$
$$= \begin{bmatrix}0\\-1\\1\end{bmatrix} - \frac{1}{2}\left\langle \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\rangle \begin{bmatrix}-1\\1\\0\end{bmatrix}$$
$$= \begin{bmatrix}0\\-1\\1\end{bmatrix} - \frac{1}{2}(0-1+0)\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

• $\underline{\text{NORMALIZE}}$: $\left\|\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}\right\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}$,

SO SET $\underset{\sim}{\vec{u}_2} = \frac{1}{\sqrt{3}/\sqrt{2}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} = \underset{\sim}{\frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}}$

$\underline{\begin{bmatrix}1\\0\\-1\end{bmatrix}}$: • $\underline{\text{SUBTRACT PROJECTIONS ONTO } \vec{u}_1 \text{ AND } \vec{u}_2}$:

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\0\\-1\end{bmatrix} - \left\langle \begin{bmatrix}1\\0\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} - \left\langle \begin{bmatrix}1\\0\\-1\end{bmatrix}, \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}\right\rangle \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$
$$= \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{1}{2}\left\langle \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\rangle \begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{2}{3}\left\langle \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}\right\rangle \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$
$$= \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{1}{2}(-1+0+0)\begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{2}{3}\left(-\frac{1}{2} + 0 - 1\right)\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$
$$= \begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} + \begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow \text{TOSS OUT!}$$

$\therefore \underset{\sim}{\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}\right\}}$ IS AN ORTHONORMAL BASIS FOR SPAN $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

14. APPLY GRAM-SCHMIDT TO $\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$:

$\begin{bmatrix}2\\1\\2\end{bmatrix}$: $\left\|\begin{bmatrix}2\\1\\2\end{bmatrix}\right\| = \sqrt{4+1+4} = \sqrt{9} = 3$, SO LET $\vec{u}_1 = \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$.

$\begin{bmatrix}1\\1\\-1\end{bmatrix}$: • SUBTRACT PROJECTION ONTO $\vec{u}_1$:

$$\begin{bmatrix}1\\1\\-1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\1\\-1\end{bmatrix} - \left\langle \begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}\right\rangle \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\-1\end{bmatrix} - \frac{1}{9}\left\langle \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\rangle \begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix} - \frac{1}{9}(2+1-2)\begin{bmatrix}2\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\-1\end{bmatrix} - \frac{1}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}7/9\\8/9\\-11/9\end{bmatrix} = \frac{1}{9}\begin{bmatrix}7\\8\\-11\end{bmatrix}$$

• NORMALIZE: $\left\|\frac{1}{9}\begin{bmatrix}7\\8\\-11\end{bmatrix}\right\| = \frac{1}{9}\left\|\begin{bmatrix}7\\8\\-11\end{bmatrix}\right\| = \frac{1}{9}\sqrt{49+64+121}$

$$= \frac{1}{9}\sqrt{234} = \frac{1}{9}\sqrt{9\cdot 26} = \frac{1}{3}\sqrt{26}$$

SO LET $\vec{u}_2 = \dfrac{1}{\frac{1}{3}\sqrt{26}} \cdot \frac{1}{9}\begin{bmatrix}7\\8\\-11\end{bmatrix} = \dfrac{1}{3\sqrt{26}}\begin{bmatrix}7\\8\\-11\end{bmatrix}$

$\begin{bmatrix}1\\0\\1\end{bmatrix}$: • SUBTRACT PROJECTIONS ONTO $\vec{u}_1, \vec{u}_2$:

$$\begin{bmatrix}1\\0\\1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\0\\1\end{bmatrix} - \left\langle \begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}\right\rangle \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} - \left\langle \begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{3\sqrt{26}}\begin{bmatrix}7\\8\\-11\end{bmatrix}\right\rangle \frac{1}{3\sqrt{26}}\begin{bmatrix}7\\8\\-11\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{9}\left\langle \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}\right\rangle \begin{bmatrix}2\\1\\2\end{bmatrix} - \frac{1}{234}\left\langle \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}7\\8\\-11\end{bmatrix}\right\rangle \begin{bmatrix}7\\8\\-11\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{9}(2+0+2)\begin{bmatrix}2\\1\\2\end{bmatrix} - \frac{1}{234}(7+0-11)\begin{bmatrix}7\\8\\-11\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{4}{9}\begin{bmatrix}2\\1\\2\end{bmatrix} + \frac{2}{117}\begin{bmatrix}7\\8\\-11\end{bmatrix} = \begin{bmatrix}3/13\\-4/13\\-1/13\end{bmatrix} = \frac{1}{13}\begin{bmatrix}3\\-4\\-1\end{bmatrix}.$$

• NORMALIZE: $\left\|\frac{1}{13}\begin{bmatrix}3\\-4\\-1\end{bmatrix}\right\| = \frac{1}{13}\left\|\begin{bmatrix}3\\-4\\-1\end{bmatrix}\right\| = \frac{1}{13}\sqrt{9+16+1} = \frac{1}{13}\sqrt{26}$,

SO LET $\vec{u}_3 = \dfrac{1}{\frac{1}{13}\sqrt{26}} \cdot \frac{1}{13}\begin{bmatrix}3\\-4\\-1\end{bmatrix} = \dfrac{1}{\sqrt{26}}\begin{bmatrix}3\\-4\\-1\end{bmatrix}$

$\therefore \left\{\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}, \frac{1}{3\sqrt{26}}\begin{bmatrix}7\\8\\-11\end{bmatrix}, \frac{1}{\sqrt{26}}\begin{bmatrix}3\\-4\\-1\end{bmatrix}\right\}$ IS AN ORTHONORMAL BASIS FOR SPAN $\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$.