1. $\vec{v}, \vec{w} \in V$; $f(q) \overset{\text{DEF}}{=} \|\vec{v} - q\vec{w}\|^2$.

(a) FOR EACH SCALAR q, f GIVES US THE [SQUARED] DISTANCE BETWEEN $\vec{v}$ AND THAT MULTIPLE OF $\vec{w}$.

(b) BECAUSE $\|\vec{v} - q\vec{w}\| \geq 0$, $f(q) \geq 0 \ \forall q$

(c) EXPANDING VIA INNER PRODUCTS,

$$\begin{aligned} f(q) = \|\vec{v} - q\vec{w}\|^2 &= \langle \vec{v} - q\vec{w}, \vec{v} - q\vec{w} \rangle \\ &= \langle \vec{v}, \vec{v} - q\vec{w} \rangle - q\langle \vec{w}, \vec{v} - q\vec{w} \rangle \\ &= \langle \vec{v}, \vec{v} \rangle - q\langle \vec{v}, \vec{w} \rangle - q\langle \vec{w}, \vec{v} \rangle + q^2 \langle \vec{w}, \vec{w} \rangle \\ &= \underbrace{\langle \vec{w}, \vec{w} \rangle q^2 - 2\langle \vec{v}, \vec{w} \rangle q + \langle \vec{v}, \vec{v} \rangle}_{\text{A QUADRATIC POLYNOMIAL IN } q!}. \end{aligned}$$

(d) [RECALL THAT $ax^2 + bx + c \geq 0 \ \forall x \Leftrightarrow D \leq 0$, WHERE $D = b^2 - 4ac$.]

$f(q)$ IS A QUADRATIC POLYNOMIAL IN q (AS IN (c)) WHICH IS ALWAYS ≥ 0 (AS IN (b)).

THUS $D \leq 0$, I.E., $(-2\langle \vec{v}, \vec{w} \rangle)^2 - 4\langle \vec{w}, \vec{w} \rangle \langle \vec{v}, \vec{v} \rangle \leq 0$

I.E., $\cancel{4}\langle \vec{v}, \vec{w} \rangle^2 \leq \cancel{4}\langle \vec{w}, \vec{w} \rangle \langle \vec{v}, \vec{v} \rangle$

SO $|\langle \vec{v}, \vec{w} \rangle| \leq \sqrt{\langle \vec{w}, \vec{w} \rangle} \cdot \sqrt{\langle \vec{v}, \vec{v} \rangle}$

I.E., $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$.

THIS IS CALLED THE CAUCHY–SCHWARZ INEQUALITY, AND IT IS TRUE FOR ANY TWO VECTORS $\vec{v}, \vec{w}$ IN ANY INNER PRODUCT SPACE!

(e) IF $\|\vec{w}\| \neq 0$, WE CAN MINIMIZE $f(q)$ BY SOLVING $f'(q) = 0$:

$$0 = f'(q) = 2\langle \vec{w}, \vec{w} \rangle q - 2\langle \vec{v}, \vec{w} \rangle \Rightarrow q = \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle}.$$

(f) THE VALUE OF q FOUND IN PART (e) TELLS US THE MULTIPLE OF $\vec{w}$ THAT IS CLOSEST TO $\vec{v}$ — WE CALL THIS MULTIPLE OF $\vec{w}$ THE PROJECTION OF $\vec{v}$ ONTO $\vec{w}$, GIVEN BY:

$$\text{PROJ}_{\vec{w}} \vec{v} \overset{\text{DEF}}{=} \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle} \vec{w}.$$

2. IF $\vec{v}, \vec{w} \neq \vec{0}$, CAUCHY–SCHWARZ TELLS US THAT $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|$,

SO THAT $\dfrac{|\langle \vec{v}, \vec{w} \rangle|}{\|\vec{v}\| \cdot \|\vec{w}\|} \leq 1$, I.E., $-1 \leq \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|} \leq 1$.

SO ARCCOS EXISTS!

WE DEFINE THE ANGLE θ BETWEEN $\vec{v}$ & $\vec{w}$ BY $\cos\theta = \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \cdot \|\vec{w}\|}$.

3. LET $\vec{v}, \vec{w} \in V$. (V: INNER PRODUCT SPACE)

(a) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \Leftrightarrow \|\vec{v} + \vec{w}\|^2 \leq (\|\vec{v}\| + \|\vec{w}\|)^2$ BECAUSE BOTH SIDES ARE ≥ 0

$\Leftrightarrow \langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$

$\Leftrightarrow \cancel{\langle \vec{v}, \vec{v} \rangle} + 2\langle \vec{v}, \vec{w} \rangle + \cancel{\langle \vec{w}, \vec{w} \rangle} \leq \cancel{\|\vec{v}\|^2} + 2\|\vec{v}\| \, \|\vec{w}\| + \cancel{\|\vec{w}\|^2}$

$\Leftrightarrow \langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\| \cdot \|\vec{w}\|$, WHICH IS TRUE BY THE CAUCHY–SCHWARZ INEQUALITY. ■

(b) TO SHOW THAT $\|\vec{v} + \vec{w}\| \geq \big| \|\vec{v}\| - \|\vec{w}\| \big|$, WE NEED TO SHOW THAT

$\|\vec{v}\| - \|\vec{w}\| \leq \|\vec{v} + \vec{w}\|$ AND $\|\vec{w}\| - \|\vec{v}\| \leq \|\vec{v} + \vec{w}\|$

GET ALL +'S TO SET UP △ INEQUALITY...

I.E., $\|\vec{v}\| \leq \|\vec{v} + \vec{w}\| + \|\vec{w}\|$ AND $\|\vec{w}\| \leq \|\vec{v} + \vec{w}\| + \|\vec{v}\|$

AND THE TRICK!

I.E., $\|(\vec{v} + \vec{w}) + (-\vec{w})\| \leq \|\vec{v} + \vec{w}\| + \|-\vec{w}\|$

AND $\|(\vec{v} + \vec{w}) + (-\vec{v})\| \leq \|\vec{v} + \vec{w}\| + \|-\vec{v}\|$,

BOTH OF WHICH ARE TRUE BY THE TRIANGLE INEQUALITY. ■

4. LET $f, g \in C([0,1])$ → AN INNER PRODUCT SPACE, VIA $\langle f,g\rangle = \int_0^1 fg$

(a) $\int_0^1 f(x)g(x)\,dx = \langle f,g\rangle$; $\int_0^1 f(x)^2\,dx = \langle f,f\rangle$; AND $\int_0^1 g(x)^2\,dx = \langle g,g\rangle$.

THUS, IN TERMS OF INNER PRODUCTS,

$$\left(\int_0^1 f(x)g(x)\,dx\right)^2 \le \left(\int_0^1 f(x)^2dx\right)\left(\int_0^1 g(x)^2dx\right)$$

JUST SAYS THAT $\langle f,g\rangle^2 \le \langle f,f\rangle\langle g,g\rangle$

TAKING SQUARE ROOTS, $|\langle f,g\rangle| \le \sqrt{\langle f,f\rangle}\cdot\sqrt{\langle g,g\rangle}$

I.E., $|\langle f,g\rangle| \le \|f\|\cdot\|g\|$,

WHICH IS TRUE BY CAUCHY-SCHWARZ.

(b) $\int_0^1 x f(x)\,dx \le \sqrt{\frac{1}{3}\int_0^1 f(x)^2dx}$ CAN BE PROVEN BY USING THE CAUCHY-SCHWARZ INEQUALITY ON $g(x) = x$ AND $f(x)$:

$$\int_0^1 x f(x)\,dx = \langle g,f\rangle \le \|g\|\cdot\|f\| = \sqrt{\langle g,g\rangle}\cdot\sqrt{\langle f,f\rangle}$$

$$= \sqrt{\int_0^1 x^2dx}\cdot\sqrt{\int_0^1 f(x)^2dx}$$

$$= \sqrt{\tfrac{1}{3}}\cdot\sqrt{\int_0^1 f(x)^2dx} \quad \blacksquare$$

(c) $\sqrt{\int_0^1 (f(x)+g(x))^2dx} = \sqrt{\langle f+g, f+g\rangle} = \|f+g\|$;

$\sqrt{\int_0^1 f(x)^2dx} = \sqrt{\langle f,f\rangle} = \|f\|$; AND $\sqrt{\int_0^1 g(x)^2dx} = \sqrt{\langle g,g\rangle} = \|g\|$,

SO $\sqrt{\int_0^1 (f(x)+g(x))^2dx} \le \sqrt{\int_0^1 f(x)^2dx} + \sqrt{\int_0^1 g(x)^2dx}$

IS JUST $\|f+g\| \le \|f\| + \|g\|$,

THE TRIANGLE INEQUALITY ON $C([0,1])$! $\blacksquare$

5. $\vec{a} \in V$ WITH $\vec{a} \ne 0$; $L: V \to V$ DEFINED BY $\vec{v} \mapsto \text{Proj}_{\vec{a}}\vec{v}$.

I.E., $L(\vec{v}) = \text{Proj}_{\vec{a}}\vec{v} = \dfrac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}$.

(a) L IS A LINEAR TRANSFORMATION:

⓪ $[L(\vec{0}) = \vec{0}]$ $L(\vec{0}) \overset{\text{DEF}}{=} \dfrac{\langle\vec{0},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} = \dfrac{0}{\langle\vec{a},\vec{a}\rangle}\vec{a} = 0\vec{a} = \vec{0}$ ✓

① $[\forall\, \vec{v} \in V$ AND $q \in \mathbb{R}$, $L(q\vec{v}) = qL(\vec{v})]$

LET $\vec{v} \in V$ AND $q \in \mathbb{R}$ BE GIVEN.

THEN $L(q\vec{v}) \overset{\text{DEF}}{=} \dfrac{\langle q\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} = q\dfrac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} = qL(\vec{v})$ ✓

② $[\forall\, \vec{v}_1, \vec{v}_2 \in V$, $L(\vec{v}_1+\vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)]$

LET $\vec{v}_1, \vec{v}_2 \in V$ BE GIVEN.

THEN $L(\vec{v}_1+\vec{v}_2) \overset{\text{DEF}}{=} \dfrac{\langle\vec{v}_1+\vec{v}_2,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}$

$$= \frac{\langle\vec{v}_1,\vec{a}\rangle + \langle\vec{v}_2,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}$$

$$= \left(\frac{\langle\vec{v}_1,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle} + \frac{\langle\vec{v}_2,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\right)\vec{a}$$

$$= \frac{\langle\vec{v}_1,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} + \frac{\langle\vec{v}_2,\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}$$

$$= L(\vec{v}_1) + L(\vec{v}_2) \checkmark$$

THUS L IS A LINEAR TRANSFORMATION $\blacksquare$

(b) $L \circ L = L$: [I.E., $\forall\, \vec{v} \in V$, $(L\circ L)(\vec{v}) = L(\vec{v})$]

LET $\vec{v} \in V$ BE GIVEN.

THEN $(L\circ L)(\vec{v}) \overset{\text{DEF}}{=} L(L(\vec{v})) = L\left(\dfrac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}\right)$

$$= \frac{\left\langle \frac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a}, \vec{a}\right\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} = \frac{\frac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\cancel{\langle\vec{a},\vec{a}\rangle}}{\cancel{\langle\vec{a},\vec{a}\rangle}}\vec{a}$$

$$= \frac{\langle\vec{v},\vec{a}\rangle}{\langle\vec{a},\vec{a}\rangle}\vec{a} = L(\vec{v}). \quad \blacksquare$$

6. $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$, $\vec{c} = \begin{bmatrix}2\\2\\2\end{bmatrix}$. (VECTORS IN $\mathbb{R}^3$, CONSIDERED AS AN INNER PRODUCT SPACE VIA THE DOT PRODUCT)

FOR REFERENCE, THE DOT PRODUCTS ARE

$\cdot$	$\vec{a}$	$\vec{b}$	$\vec{c}$
$\vec{a}$	14	2	12
$\vec{b}$	2	2	0
$\vec{c}$	12	0	12

(a) $\mathrm{PROJ}_{\vec{a}}\vec{b} = \dfrac{\langle \vec{b},\vec{a}\rangle}{\langle \vec{a},\vec{a}\rangle}\vec{a} = \dfrac{2}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1/7\\2/7\\3/7\end{bmatrix}$

NOTE THAT $\mathrm{PROJ}_{\vec{a}}\vec{b} \neq \mathrm{PROJ}_{\vec{b}}\vec{a}$

(b) $\mathrm{PROJ}_{\vec{b}}\vec{a} = \dfrac{\langle \vec{a},\vec{b}\rangle}{\langle \vec{b},\vec{b}\rangle}\vec{b} = \dfrac{2}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

(c) $\mathrm{PROJ}_{\vec{b}}\vec{c} = \dfrac{\langle \vec{c},\vec{b}\rangle}{\langle \vec{b},\vec{b}\rangle}\vec{b} = \dfrac{0}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) $\mathrm{PROJ}_{\vec{c}}\vec{b} = \dfrac{\langle \vec{b},\vec{c}\rangle}{\langle \vec{c},\vec{c}\rangle}\vec{c} = \dfrac{0}{12}\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(e) $\mathrm{PROJ}_{\vec{a}}(\mathrm{PROJ}_{\vec{b}}\vec{c}) \overset{\text{PART (c)}}{=} \mathrm{PROJ}_{\vec{a}}\begin{bmatrix}0\\0\\0\end{bmatrix} = \dfrac{\left\langle \begin{bmatrix}0\\0\\0\end{bmatrix},\vec{a}\right\rangle}{\langle \vec{a},\vec{a}\rangle}\vec{a} = \dfrac{0}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(f) $\mathrm{PROJ}_{\vec{b}}(\mathrm{PROJ}_{\vec{a}}\vec{c}) = \mathrm{PROJ}_{\vec{b}}\left(\dfrac{\langle \vec{c},\vec{a}\rangle}{\langle \vec{a},\vec{a}\rangle}\vec{a}\right) = \mathrm{PROJ}_{\vec{b}}\left(\dfrac{12}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$

$= \mathrm{PROJ}_{\vec{b}}\begin{bmatrix}6/7\\12/7\\18/7\end{bmatrix} = \dfrac{\left\langle \begin{bmatrix}6/7\\12/7\\18/7\end{bmatrix},\vec{b}\right\rangle}{\langle \vec{b},\vec{b}\rangle}\vec{b} = \dfrac{12/7}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-6/7\\0\\6/7\end{bmatrix}$

NOTE THAT $\mathrm{PROJ}_{\vec{a}} \circ \mathrm{PROJ}_{\vec{b}} \neq \mathrm{PROJ}_{\vec{b}} \circ \mathrm{PROJ}_{\vec{a}}$

7. $f(x) = 1$, $g(x) = x$, $h(x) = x^2$ IN $C([0,1])$ → INNER PRODUCT SPACE VIA $\langle f_1, f_2\rangle = \int_0^1 f_1 f_2$

FOR REFERENCE, THE INNER PRODUCTS ARE

	f	g	h
f	1	1/2	1/3
g	1/2	1/3	1/4
h	1/3	1/4	1/5

(a) $\|f\| = \sqrt{\langle f,f\rangle} = \sqrt{1} = 1$

(b) $\|g\| = \sqrt{\langle g,g\rangle} = \sqrt{1/3} = 1/\sqrt{3}$

(c) $\|h\| = \sqrt{\langle h,h\rangle} = \sqrt{1/5} = 1/\sqrt{5}$

(d) $\mathrm{PROJ}_f\, g = \dfrac{\langle g,f\rangle}{\langle f,f\rangle} f = \dfrac{1/2}{1} f = \dfrac{1}{2} f = \text{"}\dfrac{1}{2}\text{"}$

(e) $\mathrm{PROJ}_g\, f = \dfrac{\langle f,g\rangle}{\langle g,g\rangle} g = \dfrac{1/2}{1/3} g = \dfrac{3}{2} g = \text{"}\dfrac{3}{2}x\text{"}$

(f) $\mathrm{PROJ}_g\, h = \dfrac{\langle h,g\rangle}{\langle g,g\rangle} g = \dfrac{1/4}{1/3} g = \dfrac{3}{4} g = \text{"}\dfrac{3}{4}x\text{"}$

(g) $\mathrm{PROJ}_h\, g = \dfrac{\langle g,h\rangle}{\langle h,h\rangle} h = \dfrac{1/4}{1/5} h = \dfrac{5}{4} h = \text{"}\dfrac{5}{4}x^2\text{"}$

8. INNER PRODUCT SPACE $C([-\pi,\pi])$, WITH $\langle f, g\rangle \stackrel{\text{DEF}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} fg$.

$S_k(x) = \sin kx$; $f(x) = x$.

- $$\text{PROJ}_{S_1} f = \frac{\langle f, S_1\rangle}{\langle S_1, S_1\rangle} S_1 = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} x \sin x\, dx}{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} \sin^2 x\, dx} \sin x$$

$$= \frac{2\pi}{\pi} \sin x = \underline{2 \sin x}.$$

- $$\text{PROJ}_{S_2} f = \frac{\langle f, S_2\rangle}{\langle S_2, S_2\rangle} S_2 = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} x \sin 2x\, dx}{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} \sin^2 2x\, dx} \sin 2x$$

$$= \frac{-\pi}{\pi} \sin 2x = \underline{-\sin 2x}.$$

- $$\text{PROJ}_{S_3} f = \frac{\langle f, S_3\rangle}{\langle S_3, S_3\rangle} S_3 = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} x \sin 3x\, dx}{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} \sin^2 3x\, dx} \sin 3x$$

$$= \frac{\frac{2}{3}\pi}{\pi} \sin 3x = \underline{\frac{2}{3} \sin 3x}.$$

- IN GENERAL, $$\text{PROJ}_{S_k} f = \frac{\langle f, S_k\rangle}{\langle S_k, S_k\rangle} = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} x \sin kx\, dx}{\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\pi}^{\pi} \sin^2 kx\, dx} \sin kx$$

$$= \frac{\frac{2\pi}{k}(-1)^{k+1}}{\pi} \sin kx = \underline{\frac{2}{k}(-1)^{k+1} \sin kx}.$$