1. (a) IF $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ AND $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, THEN THE <u>DOT PRODUCT</u> OF $\vec{x}$ AND $\vec{y}$ IS DEFINED BY $\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n$.

(b) INTUITIVELY, $\vec{x} \cdot \vec{y}$ MEASURES THE "AGREEMENT"* OF THE VECTORS $\vec{x}$ AND $\vec{y}$: POSITIVE $\leftrightarrow$ AGREEMENT AND NEGATIVE $\leftrightarrow$ DISAGREEMENT, WITH THE SIZE OF $\vec{x} \cdot \vec{y}$ INDICATING STRENGTH. $\vec{x} \cdot \vec{y} = 0$ INDICATES THAT $\vec{x}$ AND $\vec{y}$ ARE GEOMETRICALLY INDEPENDENT OF ONE ANOTHER.

* IN THE SENSE OF TWO FORCES OR VELOCITIES.

(c) THE DOT PRODUCT IS:

- <u>BILINEAR</u>: $\begin{cases} \forall \vec{x}, \vec{y} \in \mathbb{R}^n \text{ AND SCALAR } \alpha,\ (\alpha\vec{x}) \cdot \vec{y} = \alpha(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (\alpha\vec{y}) \\ \forall \vec{x}_1, \vec{x}_2, \vec{y} \in \mathbb{R}^n,\ (\vec{x}_1 + \vec{x}_2) \cdot \vec{y} = \vec{x}_1 \cdot \vec{y} + \vec{x}_2 \cdot \vec{y} \\ \forall \vec{x}, \vec{y}_1, \vec{y}_2 \in \mathbb{R}^n,\ \vec{x} \cdot (\vec{y}_1 + \vec{y}_2) = \vec{x} \cdot \vec{y}_1 + \vec{x} \cdot \vec{y}_2 \end{cases}$
- <u>SYMMETRIC</u>: $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
- <u>POSITIVE-DEFINITE</u>: $\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \cdot \vec{x} \geq 0$, (POSITIVE)
  AND $\vec{x} \cdot \vec{x} = 0 \Leftrightarrow \vec{x} = \vec{0}$. (DEFINITE)

(d) FOR ANY i, $\vec{e}_i \cdot \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 0 + \cdots + 1 + \cdots + 0 = 1$.

IF $i \neq j$, $\vec{e}_i \cdot \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 0 + \cdots + 0 = 0$.

IN SUM, $\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \overset{\text{DEF}}{=} \begin{cases} 1 & \text{IF } i = j \\ 0 & \text{IF } i \neq j. \end{cases}$

2. (a) THE <u>NORM</u> OF $\vec{x}$ IS DEFINED BY $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$.

(b) BY THE [GENERALIZED] PYTHAGOREAN THEOREM, IF $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$ IS INTERPRETED AS THE <u>LENGTH</u> OF THE VECTOR $\vec{x}$.

(c) THE PROPERTIES OF THE NORM DERIVE FROM ITS DEFINITION IN TERMS OF THE DOT PRODUCT:

- $\forall \vec{x} \in \mathbb{R}^n$, $\|\vec{x}\| \geq 0$, AND $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$. (FROM THE DOT PRODUCT BEING POSITIVE-DEFINITE)
- $\forall \alpha \in \mathbb{R}$ AND $\vec{x} \in \mathbb{R}^n$, $\|\alpha\vec{x}\| = |\alpha| \cdot \|\vec{x}\|$.
  $\left(\|\alpha\vec{x}\| \overset{\text{DEF}}{=} \sqrt{(\alpha\vec{x}) \cdot (\alpha\vec{x})} = \sqrt{\alpha^2(\vec{x} \cdot \vec{x})} = |\alpha| \sqrt{\vec{x} \cdot \vec{x}} = |\alpha| \cdot \|\vec{x}\|\right)$

(d) $\|\vec{e}_i\| = 1$. $\left(\|\vec{e}_i\| \overset{\text{DEF}}{=} \sqrt{\vec{e}_i \cdot \vec{e}_i} = \sqrt{1} = 1\right)$

3. (a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 3 + 4 \cdot (-2) = -1 + 0 + 9 - 8 = 0$.

$\vec{a} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 1 \cdot 2 + 2(-2) + 3 \cdot 2 + 4(-2) = 2 - 4 + 6 - 8 = -4$.

$\vec{a} \cdot \vec{d} = \vec{d} \cdot \vec{a} = 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 = 1 + 0 + 0 + 0 = 1$.

$\vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{b} = -1(2) + 0(-2) + 3 \cdot 2 + (-2)(-2) = -2 + 0 + 6 + 4 = 8$.

$\vec{b} \cdot \vec{d} = \vec{d} \cdot \vec{b} = -1(1) + 0 \cdot 0 + 3 \cdot 0 + (-2)0 = -1 + 0 + 0 + 0 = -1$.

$\vec{c} \cdot \vec{d} = \vec{d} \cdot \vec{c} = 2 \cdot 1 + (-2)0 + 2 \cdot 0 + (-2) \cdot 0 = 2 + 0 + 0 + 0 = 2$.

(i) $\vec{b}$ AND $\vec{c}$ "AGREE" MOST STRONGLY.
(8 IS THE LARGEST POSITIVE DOT PRODUCT HERE)

(ii) $\vec{a}$ AND $\vec{c}$ "DISAGREE" MOST STRONGLY
(-4 IS THE LARGEST NEGATIVE DOT PRODUCT HERE)

(iii) $\vec{a}$ AND $\vec{b}$ ARE GEOMETRICALLY INDEPENDENT
($\vec{a} \cdot \vec{b} = 0$, SO THESE NEITHER AGREE NOR DISAGREE)

(b) $\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{1^2+2^2+3^2+4^2} = \sqrt{1+4+9+16} = \underset{\sim}{\sqrt{40}}$

$\|\vec{b}\| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{(-1)^2+0^2+3^2+(-2)^2} = \sqrt{1+0+9+4} = \underset{\sim}{\sqrt{14}}$

$\|\vec{c}\| = \sqrt{\vec{c}\cdot\vec{c}} = \sqrt{2^2+(-2)^2+2^2+(-2)^2} = \sqrt{4+4+4+4} = \sqrt{16} = \underset{\sim}{4}$

$\|\vec{d}\| = \sqrt{\vec{d}\cdot\vec{d}} = \sqrt{1^2+0^2+0^2+0^2} = \sqrt{1} = \underset{\sim}{1}$

(i) $\vec{a}$ IS THE LONGEST ($\sqrt{40}$ IS THE LARGEST NORM)

(ii) $\vec{d}$ IS THE SHORTEST (1 IS THE SMALLEST NORM)

4. AN <u>INNER PRODUCT</u> $\langle\cdot,\cdot\rangle$ ON A REAL VECTOR SPACE V IS A <u>POSITIVE-DEFINITE</u>, <u>SYMMETRIC</u> [<u>BILINEAR</u>] <u>2-FORM</u> ON V,

I.E., A FUNCTION MAPPING EACH PAIR OF VECTORS $\vec{v},\vec{w}\in V$ TO A REAL NUMBER DENOTED $\langle\vec{v},\vec{w}\rangle$ POSSESSING THE FOLLOWING PROPERTIES: (MIRRORING THOSE OF THE DOT PRODUCT ON $\mathbb{R}^n$)

<u>BILINEAR</u>:
$$\begin{cases} \forall\, \vec{v},\vec{w}\in V \text{ AND } \alpha\in\mathbb{R}, & \langle\alpha\vec{v},\vec{w}\rangle = \alpha\langle\vec{v},\vec{w}\rangle = \langle\vec{v},\alpha\vec{w}\rangle \\ \forall\, \vec{v}_1,\vec{v}_2,\vec{w}\in V, & \langle\vec{v}_1+\vec{v}_2,\vec{w}\rangle = \langle\vec{v}_1,\vec{w}\rangle + \langle\vec{v}_2,\vec{w}\rangle \\ \forall\, \vec{v},\vec{w}_1,\vec{w}_2\in V, & \langle\vec{v},\vec{w}_1+\vec{w}_2\rangle = \langle\vec{v},\vec{w}_1\rangle + \langle\vec{v},\vec{w}_2\rangle \end{cases}$$

<u>SYMMETRIC</u>: $\forall\, \vec{v},\vec{w}\in V,\ \langle\vec{v},\vec{w}\rangle = \langle\vec{w},\vec{v}\rangle$

<u>POSITIVE-DEFINITE</u>: $\forall\, \vec{v}\in V,\ \langle\vec{v},\vec{v}\rangle \geq 0$
AND $\langle\vec{v},\vec{v}\rangle = 0 \iff \vec{v}=\vec{0}$.

AN <u>INNER PRODUCT SPACE</u> IS A VECTOR SPACE V EQUIPPED WITH AN INNER PRODUCT ON V.

* NOTE THAT THE DOT PRODUCT ON $\mathbb{R}^n$ <u>IS</u> AN <u>INNER PRODUCT</u> ON $\mathbb{R}^n$, SO $\mathbb{R}^n$ CAN BE CONSIDERED AN INNER PRODUCT SPACE WITH THE DOT PRODUCT SERVING AS ITS INNER PRODUCT.

5. IF $[a,b]$ IS A NONTRIVIAL CLOSED INTERVAL IN $\mathbb{R}$, WE CAN DEFINE AN INNER PRODUCT ON $C([a,b])$ BY

$$\underset{\sim}{\langle f,g\rangle \overset{\text{DEF}}{=} \int_a^b fg}$$

(NOTE THAT THIS IS OFTEN SCALED BY SOME CONSTANT, E.G., $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg$ FOR $C([-\pi,\pi])$, JUST TO SIMPLIFY FORMULAE IN CERTAIN CONTEXTS.)

6. GIVEN AN INNER PRODUCT SPACE $(V,\langle\cdot,\cdot\rangle)$, WE DEFINE THE <u>NORM</u> OF EACH VECTOR $\vec{v}\in V$ BY $\underset{\sim}{\|\vec{v}\| \overset{\text{DEF}}{=} \sqrt{\langle\vec{v},\vec{v}\rangle}}$.

(MIRRORING OUR DEFINITION OF THE NORM ON $\mathbb{R}^n$)

THE NORM INHERITS TWO PROPERTIES FROM THE INNER PRODUCT USED TO DEFINE IT:

<u>POSITIVE-DEFINITE</u>: $\forall\, \vec{v}\in V,\ \|\vec{v}\|\geq 0$
AND $\|\vec{v}\| = 0 \iff \vec{v}=\vec{0}$

<u>ABSOLUTE HOMOGENEITY</u>: $\forall\, \vec{v}\in V$ AND $\alpha\in\mathbb{R}$, $\|\alpha\vec{v}\| = |\alpha|\cdot\|\vec{v}\|$.

- THE QUANTITY $\|\vec{v}-\vec{w}\|$, THE NORM OF THE VECTOR DIFFERENCE OF $\vec{v}$ AND $\vec{w}$, IS INTERPRETED AS THE <u>DISTANCE</u> BETWEEN THE VECTORS $\vec{v}$ AND $\vec{w}$.

7. IN AN INNER PRODUCT SPACE, OUR BASIC LINEAR ALGEBRA CONCEPTS ARE SUPPLEMENTED BY GEOMETRIC NOTIONS SUCH AS LENGTH OF A VECTOR, DISTANCE BETWEEN TWO VECTORS, AND ANGLES BETWEEN VECTORS.

WHEREAS BEFORE, AN INCONSISTENT LINEAR COMBINATION WAS SIMPLY INCONSISTENT, IN AN INNER PRODUCT SPACE WE CAN MAKE SENSE OF (AND ANSWER) THE QUESTION OF WHAT THE CLOSEST APPROXIMATE SOLUTION IS.

8. IF V AND W ARE INNER PRODUCT SPACES, AN ISOMETRY $\phi: V \to W$ IS AN ISOMORPHISM (BIJECTIVE L.T.) THAT ALSO RESPECTS THE INNER PRODUCTS ON V & W:

$$\boxed{\forall\, \vec{v}_1, \vec{v}_2 \in V, \quad \langle \vec{v}_1, \vec{v}_2 \rangle_V = \langle \phi(\vec{v}_1), \phi(\vec{v}_2) \rangle_W}$$

9. SETS ARE STRUCTURELESS COLLECTIONS OF OBJECTS; SO FOR TWO SETS A, B TO BE EQUIVALENT AS SETS, ALL WE NEED IS A BIJECTION $\phi: A \to B$ TO PAIR UP THEIR ELEMENTS.

IN VECTOR SPACES, WE CAN FORM LINEAR COMBINATIONS OF OUR VECTORS, SO FOR TWO VECTOR SPACES V, W TO BE EQUIVALENT AS VECTOR SPACES, WE REQUIRE OUR BIJECTION $\phi: V \to W$ TO RESPECT LINEAR COMBINATIONS, I.E., TO BE AN ISOMORPHISM.

IN INNER PRODUCT SPACES, WE CAN NOT ONLY FORM LINEAR COMBINATIONS OF OUR VECTORS, BUT ALSO FIND INNER PRODUCTS BETWEEN VECTORS. SO FOR TWO INNER PRODUCT SPACES V, W TO BE EQUIVALENT, WE REQUIRE OUR BIJECTION $\phi: V \to W$ TO RESPECT BOTH LINEAR COMBINATIONS AND INNER PRODUCTS, I.E., TO BE AN ISOMETRY.