1. A SQUARE MATRIX A IS NONSINGULAR IF $\det A \neq 0$; IT IS SINGULAR IF $\det A = 0$.

2. BECAUSE OUR BASIC ROW OPERATIONS (SWITCHING TWO ROWS, SCALING A ROW BY A NONZERO FACTOR, AND ADDING A MULTIPLE OF ONE ROW TO ANOTHER) DO NOT AFFECT WHETHER OR NOT THE DETERMINANT OF THE MATRIX IS ZERO (THEY, RESPECTIVELY, NEGATE, SCALE, OR DO NOT CHANGE THE DETERMINANT), A MATRIX A IS NONSINGULAR $\Leftrightarrow$ ITS REDUCED FORM IS NONSINGULAR.

- IF A GIVES A PIVOT IN EVERY COLUMN, THEN A REDUCES TO THE IDENTITY MATRIX, WHICH IS NONSINGULAR BECAUSE $\det I = 1 \neq 0$.
- IF NOT, THEN THE COLUMNS OF A ARE NOT LINEARLY INDEPENDENT, SO ONE COLUMN (ANY NON-PIVOT COLUMN) CAN BE WRITTEN AS A LINEAR COMBINATION OF THE REST; WE CAN THEN USE MULTILINEARITY OF DET TO SHOW THAT $\det A = 0$, AND THUS A IS SINGULAR.

3. USING THE FACT THAT A SQUARE MATRIX A IS NONSINGULAR $\Leftrightarrow$ A GIVES A PIVOT IN EVERY ROW + COLUMN:

(a) A IS NONSINGULAR $\Leftrightarrow$ THE L.T. A IS BIJECTIVE, AND THUS INVERTIBLE

($\therefore$ A IS SINGULAR $\Leftrightarrow$ A IS NEITHER INJECTIVE NOR SURJECTIVE NOR INVERTIBLE)

(b) A IS NONSINGULAR $\Leftrightarrow$ THE COLUMNS OF A FORM A BASIS FOR $\mathbb{R}^n$

($\therefore$ A IS SINGULAR $\Leftrightarrow$ THE COLUMNS OF A ARE LINEARLY DEPENDENT AND DO NOT SPAN $\mathbb{R}^n$)

(c) A IS NONSINGULAR $\Leftrightarrow$ $\ker A = \{\vec{0}\}$ AND $\operatorname{im} A = \mathbb{R}^n$ (BIJECTIVE!)

($\therefore$ A IS SINGULAR $\Leftrightarrow$ $\ker A \neq \{\vec{0}\}$ AND $\operatorname{im} A \neq \mathbb{R}^n$)

(d) A IS NONSINGULAR $\Leftrightarrow$ $\operatorname{nullity} A = 0$ AND $\operatorname{rank} A = n$ (SEE PART (c))

($\therefore$ A IS SINGULAR $\Leftrightarrow$ $\operatorname{nullity} A \neq 0$ AND $\operatorname{rank} A \neq n$)

[THE DETERMINANT LETS US DETERMINE A GREAT DEAL ABOUT A SQUARE MATRIX A!!!]

4. (a) CLAIM: IF A IS NONSINGULAR AND B IS NONSINGULAR, THEN AB IS NONSINGULAR.

PROOF: SUPPOSE THAT A IS NONSINGULAR AND B IS NONSINGULAR, I.E., $\det A \neq 0$ AND $\det B \neq 0$.

THEN $\det(AB) = \det A \cdot \det B \neq 0$,

SO BY DEFINITION, AB IS NONSINGULAR. ■

(b) CLAIM: IF AB IS SINGULAR, THEN A IS SINGULAR OR B IS SINGULAR.

PROOF: (THIS IS JUST THE CONTRAPOSITIVE OF PART (a))

SUPPOSE THAT AB IS SINGULAR, I.E., $\det(AB) = 0$.

THEN $0 = \det(AB) = \det A \cdot \det B$,

SO $\det A = 0$ OR $\det B = 0$, I.E., A IS SINGULAR OR B IS SINGULAR. ■

(c) CLAIM: IF A IS SINGULAR OR B IS SINGULAR, THEN AB IS SINGULAR.

PROOF:
- IF A IS SINGULAR, THEN $\det A = 0$, SO $\det(AB) = \det A \cdot \det B = 0 \cdot \det B = 0$; THUS, BY DEFINITION, AB IS SINGULAR. ✓
- OTHERWISE, B IS SINGULAR, I.E., $\det B = 0$, SO $\det(AB) = \det A \cdot \det B = \det A \cdot 0 = 0$; THUS, BY DEFINITION, AB IS SINGULAR. ✓ ■

[A: n×n MATRIX]

5. THE CHARACTERISTIC POLYNOMIAL OF A IS $\det(A-\lambda I)$.

(COMPUTED JUST AS ANY OTHER DETERMINANT, THIS IS A POLYNOMIAL IN THE VARIABLE λ.)

6. THE ROOTS OF THE CHARACTERISTIC POLYNOMIAL OF A ARE THE VALUES λ FOR WHICH $\det(A-\lambda I)=0$,
I.E., THE VALUES λ FOR WHICH $A-\lambda I$ IS SINGULAR,
I.E., THE VALUES λ FOR WHICH $N(A-\lambda I)\neq\{\vec{0}\}$,
I.E., THE EIGENVALUES OF A.

(THUS, WE CAN FIND THE EIGENVALUES OF A SIMPLY BY SOLVING $\det(A-\lambda I)=0$ FOR λ.)

- OUR FIELD OF SCALARS TELLS US WHAT ROOTS OF $\det(A-\lambda I)$ ARE ALLOWED, AND THUS WHAT EIGENVALUES ARE ALLOWED.

E.G., IF $\det(A-\lambda I)=\lambda^2+1$, THEN:

- OVER $\mathbb{R}$, THERE ARE NO EIGENVALUES
- OVER $\mathbb{C}$, THE VALUES $\pm i$ ARE EIGENVALUES
- OVER $\mathbb{Z}_2$, THE VALUE 1 IS AN EIGENVALUE

⋮
ETC.

7. THE CHARACTERISTIC POLYNOMIAL THUS AFFORDS US THE FOLLOWING METHOD FOR DIAGONALIZING AN $n\times n$ MATRIX A:

- FIND THE EIGENVALUES OF A (BY SOLVING $\det(A-\lambda I)=0$ FOR λ).
- FOR EACH EIGENVALUE λ, FIND A BASIS FOR $N(A-\lambda I)$ TO DETERMINE A MAXIMAL COLLECTION OF L.I. EIGENVECTORS FOR THE EIGENVALUE λ.
- IF n EIGENVECTORS ARE OBTAINED, PUT THEM INTO A MATRIX Σ AND PUT THE CORRESPONDING EIGENVALUES INTO A DIAGONAL MATRIX Λ; THEN $A=\Sigma\Lambda\Sigma^{-1}$ BY DIRECT COMPUTATION.

(IF n EIGENVECTORS ARE NOT OBTAINED, THEN A IS NOT DIAGONALIZABLE.)

* THIS METHOD CAN BE APPLIED TO ENDOMORPHISMS OF FINITE-DIMENSIONAL VECTOR SPACES BY FIRST CHOOSING ANY BASIS FOR THE VECTOR SPACE, THEN DIAGONALIZING THE RESULTING MATRIX:

$$\left(\begin{matrix} L: V\to V \\ \text{BASIS } \mathcal{V} \text{ FOR } V\end{matrix}\right) \rightsquigarrow [L]_{\mathcal{V}} = A = \Sigma\Lambda\Sigma^{-1};$$

I.E., $[\mathcal{V}]^{-1}L[\mathcal{V}] = \Sigma\Lambda\Sigma^{-1}$;

SO $L = ([\mathcal{V}]\Sigma)\,\Lambda\,(\Sigma^{-1}[\mathcal{V}]^{-1})$

$= (\underbrace{[\mathcal{V}]\Sigma}_{\text{L.C. OF THE ORIGINAL BASIS } \mathcal{V}})\,\Lambda\,([\mathcal{V}]\Sigma)^{-1}$

8. (a) $A=\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}$: • CHAR. POLY. $= \det\begin{bmatrix}-\lambda&1\\1&-\lambda\end{bmatrix} = \lambda^2-1$;

$$0=\lambda^2-1=(\lambda+1)(\lambda-1) \Rightarrow \underbrace{\lambda=-1,1}_{\text{EIGENVALUES OF } A}$$

• EIGENVECTORS FOR $\lambda=-1$: $\begin{array}{c}\begin{matrix}x_1&\;x_2\end{matrix}\\ \begin{bmatrix}1&1\\1&1\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\underbrace{\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

• EIGENVECTORS FOR $\lambda=1$: $\begin{array}{c}\begin{matrix}x_1&\;x_2\end{matrix}\\ \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

LET $X = \begin{array}{c}\begin{matrix}\lambda=-1&\lambda=1\\ \downarrow&\downarrow\end{matrix}\\ \begin{bmatrix}-1&1\\1&1\end{bmatrix}\end{array}$ AND $\Lambda=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$; THEN $A=X\Lambda X^{-1}$.

(b) $A=\underbrace{\begin{bmatrix}13&18\\-8&-11\end{bmatrix}}$: • CHAR. POLY. $= \det\begin{bmatrix}13-\lambda&18\\-8&-11-\lambda\end{bmatrix} = \dots = \lambda^2-2\lambda+1$

$$0=\lambda^2-2\lambda+1=(\lambda-1)^2 \Rightarrow \underbrace{\lambda=1}_{\text{EIGENVALUE OF } A}$$

• EIGENVECTORS FOR $\lambda=1$: $\begin{array}{c}\begin{matrix}x_1&\;x_2\end{matrix}\\ \begin{bmatrix}12&18\\-8&-12\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\underbrace{\begin{bmatrix}-3/2\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

$*$ ONLY ONE EIGENVECTOR FOUND ∴ NO EIGENBASIS, SO A IS NOT DIAGONALIZABLE.

(c) $A=\underbrace{\begin{bmatrix}32&-18\\45&-25\end{bmatrix}}$: • CHAR. POLY. $= \det\begin{bmatrix}32-\lambda&-18\\45&-25-\lambda\end{bmatrix} = \dots = \lambda^2-7\lambda+10$

$$0=\lambda^2-7\lambda+10=(\lambda-2)(\lambda-5) \Rightarrow \underbrace{\lambda=2,5}_{\text{EIGENVALUES OF } A}$$

• EIGENVECTORS FOR $\lambda=2$: $\begin{array}{c}\begin{matrix}x_1&\;x_2\end{matrix}\\ \begin{bmatrix}30&-18\\45&-27\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\underbrace{\begin{bmatrix}3/5\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

• EIGENVECTORS FOR $\lambda=5$: $\begin{array}{c}\begin{matrix}x_1&\;x_2\end{matrix}\\ \begin{bmatrix}27&-18\\45&-30\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\underbrace{\begin{bmatrix}2/3\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

LET $X = \begin{array}{c}\begin{matrix}\lambda=2&\lambda=5\\ \downarrow&\downarrow\end{matrix}\\ \begin{bmatrix}3/5&2/3\\1&1\end{bmatrix}\end{array}$ AND $\Lambda=\begin{bmatrix}2&0\\0&5\end{bmatrix}$; THEN $A=X\Lambda X^{-1}$.

(d) $A=\underbrace{\begin{bmatrix}2&-2&0\\-24&8&8\\36&-16&-10\end{bmatrix}}$: • CHAR. POLY. $= \det\begin{bmatrix}2-\lambda&-2&0\\-24&8-\lambda&8\\36&-16&-10-\lambda\end{bmatrix}$

$$= \dots = 4\lambda-\lambda^3$$

$$0=4\lambda-\lambda^3=\lambda(4-\lambda^2)=\lambda(2+\lambda)(2-\lambda)$$

$$\Rightarrow \underbrace{\lambda=0,-2,2}_{\text{EIGENVALUES OF } A}$$

• EIGENVECTORS FOR $\lambda=0$: $\begin{array}{c}\begin{matrix}x_1&\;x_2&\;x_3\end{matrix}\\ \begin{bmatrix}2&-2&0\\-24&8&8\\36&-16&-10\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\underbrace{\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

• EIGENVECTORS FOR $\lambda=-2$: $\begin{array}{c}\begin{matrix}x_1&\;x_2&\;x_3\end{matrix}\\ \begin{bmatrix}4&-2&0\\-24&10&8\\36&-16&-8\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\underbrace{\begin{bmatrix}2\\4\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

• EIGENVECTORS FOR $\lambda=2$: $\begin{array}{c}\begin{matrix}x_1&\;x_2&\;x_3\end{matrix}\\ \begin{bmatrix}0&-2&0\\-24&6&8\\36&-16&-12\end{bmatrix}\end{array} \rightsquigarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underbrace{\begin{bmatrix}1/3\\0\\1\end{bmatrix}}_{\text{EIGENVECTOR}}$

LET $X = \begin{array}{c}\begin{matrix}\lambda=0&\lambda=-2&\lambda=2\\ \downarrow&\downarrow&\downarrow\end{matrix}\\ \begin{bmatrix}1/2&2&1/3\\1/2&4&0\\1&1&1\end{bmatrix}\end{array}$ AND $\Lambda=\begin{bmatrix}0&0&0\\0&-2&0\\0&0&2\end{bmatrix}$; THEN $A=X\Lambda X^{-1}$.

(e) $A = \begin{bmatrix} -4 & -3 & 5 & -2 \\ -11 & -8 & 13 & -6 \\ -7 & -5 & 8 & -4 \\ 8 & 6 & -10 & -4 \end{bmatrix}$:

- CHAR. POLY. = DET $\begin{bmatrix} -4-\lambda & -3 & 5 & -2 \\ -11 & -8-\lambda & 13 & -6 \\ -7 & -5 & 8-\lambda & -4 \\ 8 & 6 & -10 & -4-\lambda \end{bmatrix} = \cdots = \lambda^4 - \lambda^2$

$$0 = \lambda^4 - \lambda^2 = \lambda^2(\lambda^2 - 1) = \lambda^2(\lambda+1)(\lambda-1)$$

$$\Rightarrow \lambda = \underbrace{0, -1, 1}_{\text{EIGENVALUES OF } A}$$

- EIGENVECTORS FOR $\lambda = 0$: $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -4 & -3 & 5 & -2 \\ -11 & -8 & 13 & -6 \\ -7 & -5 & 8 & -4 \\ 8 & 6 & -10 & 4 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{EIGENVECTORS}}$

- EIGENVECTORS FOR $\lambda = -1$: $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -3 & -3 & 5 & -2 \\ -11 & -7 & 13 & -6 \\ -7 & -5 & 9 & -4 \\ 8 & 6 & -10 & 5 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} -1/2 \\ -1 \\ -1/2 \\ 1 \end{bmatrix}}_{\text{EIGENVECTOR}}$

- EIGENVECTORS FOR $\lambda = 1$: $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} -5 & -3 & 5 & -2 \\ -11 & -9 & 13 & -6 \\ -7 & -5 & 7 & -4 \\ 8 & 6 & -10 & 3 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} -1/2 \\ -3/2 \\ -1 \\ 1 \end{bmatrix}}_{\text{EIGENVECTOR}}$

LET $X = \begin{array}{c} \begin{array}{ccc} \overbrace{\quad\quad\quad}^{\lambda=0} & \lambda=-1 & \lambda=1 \end{array} \\ \begin{bmatrix} -1 & -2 & -1/2 & -1/2 \\ 3 & 2 & -1 & -3/2 \\ 1 & 0 & -1/2 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$ AND $\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

THEN $A = X \Lambda X^{-1}$!