(V: VECTOR SPACE OVER A FIELD F;
$\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$: FINITE COLLECTION OF VECTORS IN V)

1. THE <u>SPAN OF $\mathcal{C}$</u> IS THE COLLECTION OF ALL VECTORS THAT CAN BE WRITTEN AS A LINEAR COMBINATION OF VECTORS OF $\mathcal{C}$; FORMALLY,

$$\mathrm{SPAN}(\mathcal{C}) = \{\underbrace{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n}_{\text{THE GENERAL FORM OF ANY L.C. OF ANY VECTORS IN } \mathcal{C}} : a_1, \ldots, a_n \in F\}$$

SO, $\vec{v} \in \mathrm{SPAN}(\mathcal{C})$ MEANS $\boxed{\exists\, a_1, \ldots, a_n \in F \text{ SO THAT } \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n}$

2. FOR $\mathcal{C}$ TO "<u>SPAN V</u>" MEANS THAT $\mathrm{SPAN}(\mathcal{C}) = V$, I.E., $\forall\, \vec{v} \in V$, $\vec{v} \in \mathrm{SPAN}(\mathcal{C})$.

THIS TELLS US THAT EVERY VECTOR OF V CAN BE WRITTEN AS A L.C. OF VECTORS IN $\mathcal{C}$. !

3. $\mathrm{SPAN}(\mathcal{C})$ IS AUTOMATICALLY A <u>SUBSPACE OF V</u> — WE CAN CHECK THIS VIA THE 3-POINT CHECKLIST FOR SUBSPACES.

($\vec{0}$ IS THE L.C. OF NO VECTORS, AND A LITTLE ALGEBRA SHOWS THAT SCALING OR ADDING L.C.'S OF VECTORS IN $\mathcal{C}$ GIVES ANOTHER L.C. OF VECTORS IN $\mathcal{C}$)

4. $\mathcal{C}, \mathcal{D}$: COLLECTIONS OF VECTORS IN V

(a) TO PROVE THAT $\mathrm{SPAN}(\mathcal{C}) \supset \mathrm{SPAN}(\mathcal{D})$, WE NEED ONLY SHOW THAT $\vec{v} \in \mathcal{D} \Rightarrow \vec{v} \in \mathrm{SPAN}(\mathcal{C})$ — A PRIORI, THIS JUST SHOWS THAT $\mathcal{D} \subset \mathrm{SPAN}(\mathcal{C})$; BUT WE ALREADY SHOWED* THAT IF $\mathcal{D} \subset \mathrm{SPAN}(\mathcal{C})$, THEN $\mathrm{SPAN}(\mathcal{D})$ IS A SUBSPACE OF $\mathrm{SPAN}(\mathcal{C})$. (* SEE PROBLEM #3)

(b) TO SHOW THAT $\mathrm{SPAN}(\mathcal{C}) = \mathrm{SPAN}(\mathcal{D})$, WE JUST SHOW "$\subset$" + "$\supset$" — I.E., ① $\vec{v} \in \mathcal{C} \Rightarrow \vec{v} \in \mathrm{SPAN}(\mathcal{D})$ AND ② $\vec{v} \in \mathcal{D} \Rightarrow \vec{v} \in \mathrm{SPAN}(\mathcal{C})$

(FOR SETS, $A = B$ IF, AND ONLY IF, $A \subset B$ AND $A \supset B$)

5. WITHOUT CHANGING THE SPAN OF A COLLECTION $\mathcal{C}$, WE CAN:

- <u>INSERT</u> ANY ELEMENT OF $\mathrm{SPAN}(\mathcal{C})$.
  (I.E., WE CAN INSERT ANY L.C. OF THE VECTORS ALREADY IN $\mathcal{C}$)
- <u>REMOVE</u> ANY $\vec{v} \in \mathcal{C}$ THAT'S IN THE SPAN OF <u>THE REST</u>.
  (I.E., WE CAN REMOVE ANY $\vec{v} \in \mathcal{C}$ THAT CAN BE WRITTEN AS AN L.C. OF THE REST — IT'S SUPERFLUOUS FOR THE SPAN!)
- <u>REPLACE</u> ANY VECTOR $\vec{v}$ IN $\mathcal{C}$ BY ANY L.C. OF $\mathcal{C}$ HAVING A <u>NONZERO</u> COEFFICIENT OF $\vec{v}$.

  E.G., $\mathrm{SPAN}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathrm{SPAN}\{4\vec{v}_1 + \vec{v}_2 - 3\vec{v}_3, \vec{v}_2, \vec{v}_3\}$
  NONZERO COEFFICIENT OF $\vec{v}_1$

(WE CAN PROVE THESE ASSERTIONS VIA THE DEFINITION OF SPAN)

6. $\mathcal{C}$: COLLECTION OF <u>COLUMN VECTORS</u> IN $\mathbb{R}^m$

(a) IN TERMS OF THE LINEAR SYSTEMS ARISING FROM $\mathcal{C}$, $\mathrm{SPAN}(\mathcal{C})$ TELLS US WHICH TARGET VECTORS GIVE <u>CONSISTENT</u> SYSTEMS!

(WHICH TARGET VECTORS CAN BE WRITTEN AS L.C.'S OF THE VECTORS OF $\mathcal{C}$? THOSE THAT ARE L.C.'S OF VECTORS OF $\mathcal{C}$ — I.E., THOSE IN $\mathrm{SPAN}(\mathcal{C})$!)

(b) IF $\mathcal{D}$ IS ANOTHER COLLECTION OF COLUMN VECTORS, WE CAN SET UP A <u>MULTIPLY-AUGMENTED MATRIX</u> $[\mathcal{C} \mid \mathcal{D}]$ SIMULTANEOUSLY CHECK WHETHER ALL VECTORS OF $\mathcal{D}$ ARE IN THE SPAN OF $\mathcal{C}$ — IF EACH AUGMENTED COLUMN GIVES A CONSISTENT SYSTEM, THEN ALL VECTORS OF $\mathcal{D}$ ARE IN $\mathrm{SPAN}(\mathcal{C})$, SO $\mathrm{SPAN}(\mathcal{C}) \supset \mathrm{SPAN}(\mathcal{D})$. (SEE PROBLEM 4(a))

WE CAN CHECK WHETHER $\mathrm{SPAN}(\mathcal{C}) = \mathrm{SPAN}(\mathcal{D})$ BY SETTING UP THE MULTIPLY-AUGMENTED SYSTEMS $[\mathcal{C} \mid \mathcal{D}]$ AND $[\mathcal{D} \mid \mathcal{C}]$ AND CHECKING BOTH FOR CONSISTENCY.

? ?

$\mathrm{SPAN}(\mathcal{C}) = \mathrm{SPAN}(\mathcal{D})$ IF, AND ONLY IF, $\mathrm{SPAN}(\mathcal{C}) \supset \mathrm{SPAN}(\mathcal{D})$ AND $\mathrm{SPAN}(\mathcal{D}) \supset \mathrm{SPAN}(\mathcal{C})$

(c) TO CHECK THAT A COLLECTION $\mathcal{C}$ OF VECTORS FROM $\mathbb{R}^m$ SPANS $\mathbb{R}^m$, ALL THAT WE NEED TO DO IS FORM THE MATRIX $[\mathcal{C}]$ (UNAUGMENTED) AND REDUCE — IF THERE IS A <u>PIVOT IN EVERY ROW</u>, THEN $\mathcal{C}$ SPANS $\mathbb{R}^m$.

WHY? WELL, THERE BEING A PIVOT IN EVERY ROW MEANS THAT FOR ANY $\vec{v} \in \mathbb{R}^m$, THE SYSTEM $[\mathcal{C} \mid \vec{v}]$ WILL BE <u>CONSISTENT</u>.
$\therefore$ EVERY $\vec{v} \in \mathbb{R}^m$ CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$
$\therefore$ $\mathrm{SPAN}(\mathcal{C}) = \mathbb{R}^m$, I.E., $\mathcal{C}$ SPANS $\mathbb{R}^m$!

7. (a) IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SPANS V, (!) THEN $\{\vec{v}_1 - 4\vec{v}_2, \vec{v}_2, \vec{v}_3\}$ ALSO SPANS V. (?)

- ON PRINCIPLE: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SPANS V

  $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 - 4\vec{v}_2\}$ SPANS V

  ↳ L.C. OF THE REST, SO IT CAN BE INSERTED

  $\Rightarrow \{\vec{v}_2, \vec{v}_3, \vec{v}_1 - 4\vec{v}_2\}$ SPANS V ∎

  ($\vec{v}_1 = (\vec{v}_1 - 4\vec{v}_2) + 4\vec{v}_2$ IS A L.C. OF THE REST, SO IT CAN BE REMOVED)

  *NOTE THAT THE ORDER OF THE VECTORS DOESN'T MATTER.

- DIRECTLY: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ SPANS V.

  I.E., $\forall \vec{v} \in V$, $\exists$ SCALARS $\alpha_1, \alpha_2, \alpha_3$ WITH $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$. (!)

  LET $\vec{v} \in V$ BE GIVEN.

  WE NEED TO SHOW THAT: $\forall \vec{v} \in V$, $\exists$ SCALARS $\beta_1, \beta_2, \beta_3$ WITH $\vec{v} = \beta_1(\vec{v}_1 - 4\vec{v}_2) + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$ (?)

  THEN BY HYPOTHESIS,
  $\exists$ SCALARS $\alpha_1, \alpha_2, \alpha_3$ WITH $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$

  TO SHOW $\exists$ $\beta_1, \beta_2, \beta_3$ AS IN (?), DO SOME SCRATCH WORK:

  $$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \vec{v} = \beta_1(\vec{v}_1 - 4\vec{v}_2) + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$$
  $$= \beta_1\vec{v}_1 + (\beta_2 - 4\beta_1)\vec{v}_2 + \beta_3\vec{v}_3$$

  ...SO WE WANT $\beta_1 = \alpha_1$, $\beta_2 - 4\beta_1 = \alpha_2$, $\beta_3 = \alpha_3$

  SOLVING, $\beta_2 = \alpha_2 + 4\beta_1 = \alpha_2 + 4\alpha_1$!

  LET $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 + 4\alpha_1$, AND $\beta_3 = \alpha_3$.

  THEN $\beta_1(\vec{v}_1 - 4\vec{v}_2) + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$

  $$= \alpha_1(\vec{v}_1 - 4\vec{v}_2) + (\alpha_2 + 4\alpha_1)\vec{v}_2 + \alpha_3\vec{v}_3$$
  $$= \alpha_1\vec{v}_1 - \cancel{4\alpha_1\vec{v}_2} + \alpha_2\vec{v}_2 + \cancel{4\alpha_1\vec{v}_2} + \alpha_3\vec{v}_3$$
  $$= \vec{v}, \text{ BY HYPOTHESIS} \quad ∎$$

(b) IF $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ AND $\{\vec{v}_1, \vec{v}_2\}$ BOTH SPAN V, (!)
THEN $\vec{v}_3$ MUST BE A LINEAR COMBINATION OF $\{\vec{v}_1, \vec{v}_2\}$ (?)

- ON PRINCIPLE: WELL, $\vec{v}_3 \in V$, SO IF $\{\vec{v}_1, \vec{v}_2\}$ SPANS V, $\vec{v}_3$ MUST BE IN THE SPAN OF $\{\vec{v}_1, \vec{v}_2\}$, I.E., $\vec{v}_3$ MUST BE A L.C. OF $\{\vec{v}_1, \vec{v}_2\}$. ∎

- DIRECTLY: SUPPOSE THAT $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ AND $\{\vec{v}_1, \vec{v}_2\}$ BOTH SPAN V.

  SINCE $\{\vec{v}_1, \vec{v}_2\}$ SPANS V,

  $\forall \vec{v} \in V$, $\exists$ SCALARS α_1, α_2 WITH $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$.

  $\vec{v}_3 \in V$, SO WE CAN APPLY THIS WITH $\vec{v} = \vec{v}_3$:

  $\exists$ SCALARS α_1, α_2 WITH $\vec{v}_3 = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$

  — THIS IS THE DEFINITION OF $\vec{v}_3 \in \text{SPAN}\{\vec{v}_1, \vec{v}_2\}$! ∎

(c) IF $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ SPANS V AND $\vec{v}_4 \in V$, (!) THEN $\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$ SPANS V. (?)

- ON PRINCIPLE: $\vec{v}_4 \in V = \text{SPAN}\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$, BY HYPOTHESIS

  — BUT THEN, WE CAN INSERT ANY VECTOR IN THIS SPAN INTO THIS COLLECTION WITHOUT CHANGING ITS SPAN, SO $\text{SPAN}\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\} = V$ AS WELL. ■

- DIRECTLY: BY HYPOTHESIS, $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ SPANS V, SO $\forall\, \vec{v}\in V$, $\exists$ SCALARS $\alpha_1,\alpha_2,\alpha_3$ WITH $\vec{v}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3$. (!)

  TO SHOW THIS SPANS V, WE NEED TO SHOW THAT:

  $\forall\, \vec{v}\in V$, $\exists$ SCALARS $\beta_1,\beta_2,\beta_3,\beta_4$ (?)
  WITH $\vec{v}=\beta_1\vec{v}_1+\beta_2\vec{v}_2+\beta_3\vec{v}_3+\beta_4\vec{v}_4$

  LET $\vec{v}\in V$ BE GIVEN. THEN BY HYPOTHESIS, $\exists$ SCALARS $\alpha_1,\alpha_2,\alpha_3$ WITH $\vec{v}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3$.

  TO FIND $\beta_1,\beta_2,\beta_3,\beta_4$, COMBINE (!) + (?):

  $\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3 = \vec{v} = \beta_1\vec{v}_1+\beta_2\vec{v}_2+\beta_3\vec{v}_3+\beta_4\vec{v}_4$

  — JUST TAKE $\beta_1=\alpha_1$, $\beta_2=\alpha_2$, $\beta_3=\alpha_3$, $\beta_4=0$!

  TAKE $\beta_1=\alpha_1$, $\beta_2=\alpha_2$, $\beta_3=\alpha_3$, $\beta_4=0$.

  THEN $\beta_1\vec{v}_1+\beta_2\vec{v}_2+\beta_3\vec{v}_3+\beta_4\vec{v}_4 = \alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\alpha_3\vec{v}_3 + \cancel{0\vec{v}_4} = \vec{v}$,

  SO $\vec{v}\in \text{SPAN}\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$, BY DEFINITION ■

(d) THE SPAN OF $\{\vec{v},\vec{w}\}$ IS THE SAME AS THE SPAN OF $\{2\vec{v}-\vec{w},\ \vec{v}+3\vec{w}\}$

- ON PRINCIPLE: $\text{SPAN}\{\vec{v},\vec{w}\} = \text{SPAN}\{\vec{v},\vec{w},\underline{2\vec{v}-\vec{w}},\underline{\vec{v}+3\vec{w}}\}$

  CAN INSERT THESE BECAUSE THEY'RE IN THE SPAN OF $\{\vec{v},\vec{w}\}$

  $= \text{SPAN}\{\cancel{\vec{v}},\vec{w},2\vec{v}-\vec{w},\vec{v}+3\vec{w}\}$

  ↳ CAN REMOVE THIS, BECAUSE IT'S IN THE SPAN OF THE REST: $\vec{v} = (\vec{v}+3\vec{w}) - 3\vec{w}$

  $= \text{SPAN}\{\cancel{\vec{w}}, 2\vec{v}-\vec{w}, \vec{v}+3\vec{w}\}$ ■

  ↳ CAN REMOVE THIS, BECAUSE IT'S IN THE SPAN OF THE REST:

  $$\vec{w} = -\frac{1}{7}\left[(2\vec{v}-\vec{w}) - 2(\vec{v}+3\vec{w})\right]$$

  (SOLVE FOR $\vec{w}$ AS A L.C. OF THE OTHER TWO)

- DIRECTLY:

  ① $\text{SPAN}\{\vec{v},\vec{w}\} \supset \text{SPAN}\{2\vec{v}-\vec{w},\vec{v}+3\vec{w}\}$:

  (NEED TO SHOW $\vec{y}\in\text{SPAN}\{2\vec{v}-\vec{w},\vec{v}+3\vec{w}\} \Rightarrow \vec{y}\in\text{SPAN}\{\vec{v},\vec{w}\}$)

  SUPPOSE THAT $\vec{y}\in \text{SPAN}\{2\vec{v}-\vec{w},\vec{v}+3\vec{w}\}$.

  BY DEFINITION, $\exists$ SCALARS α_1,α_2 WITH

  $$\vec{y} = \alpha_1(2\vec{v}-\vec{w})+\alpha_2(\vec{v}+3\vec{w}) = (2\alpha_1+\alpha_2)\vec{v} + (-\alpha_1+3\alpha_2)\vec{w}$$

  LET $\beta_1 = 2\alpha_1+\alpha_2$ AND $\beta_2 = -\alpha_1+3\alpha_2$;

  THEN $\vec{y} = \beta_1\vec{v}+\beta_2\vec{w}$, SO BY DEFINITION, $\vec{y}\in\text{SPAN}\{\vec{v},\vec{w}\}$. ✓

② SPAN$\{2\vec{v}-\vec{w}, \vec{v}+3\vec{w}\} \supset$ SPAN$\{\vec{v},\vec{w}\}$:

(NEED TO SHOW THAT $\vec{y} \in$ SPAN$\{\vec{v},\vec{w}\}$ $\Rightarrow \vec{y} \in$ SPAN$\{2\vec{v}-\vec{w}, \vec{v}+3\vec{w}\}$)

SUPPOSE THAT $\vec{y} \in$ SPAN$\{\vec{v},\vec{w}\}$.

BY DEFINITION, $\exists$ SCALARS α_1, α_2 WITH $\vec{y} = \alpha_1\vec{v} + \alpha_2\vec{w}$ (!)

TO SHOW $\vec{y} \in$ SPAN$\{2\vec{v}-\vec{w}, \vec{v}+3\vec{w}\}$, WE NEED TO FIND SCALARS β_1, β_2 WITH $\vec{y} = \beta_1(2\vec{v}-\vec{w}) + \beta_2(\vec{v}+3\vec{w})$. (?)

TO FIND β_1, β_2, COMBINE (!) AND (?) AND SOLVE FOR β_1, β_2:

$$\alpha_1\vec{v} + \alpha_2\vec{w} = \beta_1(2\vec{v}-\vec{w}) + \beta_2(\vec{v}+3\vec{w})$$
$$= (2\beta_1 + \beta_2)\vec{v} + (-\beta_1 + 3\beta_2)\vec{w}$$

SO WE WANT $2\beta_1 + \beta_2 = \alpha_1$, $-\beta_1 + 3\beta_2 = \alpha_2$:

$$\text{SCALE } \tfrac{1}{2} \rightarrow \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \left[\begin{array}{cc|c} 2 & 1 & \alpha_1 \\ -1 & 3 & \alpha_2 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2}\alpha_1 \\ -1 & 3 & \alpha_2 \end{array}\right] {+1}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2}\alpha_1 \\ 0 & \frac{7}{2} & \alpha_2 + \frac{1}{2}\alpha_1 \end{array}\right] \leftarrow \text{SCALE } \tfrac{2}{7} \rightsquigarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2}\alpha_1 \\ 0 & 1 & \frac{2}{7}\alpha_2 + \frac{1}{7}\alpha_1 \end{array}\right] {-\tfrac{1}{2}}$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{7}\alpha_1 - \frac{1}{7}\alpha_2 \\ 0 & 1 & \frac{2}{7}\alpha_2 + \frac{1}{7}\alpha_1 \end{array}\right] \Longrightarrow \begin{array}{l} \beta_1 = \frac{3}{7}\alpha_1 - \frac{1}{7}\alpha_2 \\ \beta_2 = \frac{2}{7}\alpha_2 + \frac{1}{7}\alpha_1 \end{array}$$

LET $\beta_1 = \frac{3}{7}\alpha_1 - \frac{1}{7}\alpha_2$, $\beta_2 = \frac{2}{7}\alpha_2 + \frac{1}{7}\alpha_1$.

THEN $\beta_1(2\vec{v}-\vec{w}) + \beta_2(\vec{v}+3\vec{w})$

$$= \left(\tfrac{3}{7}\alpha_1 - \tfrac{1}{7}\alpha_2\right)(2\vec{v}-\vec{w}) + \left(\tfrac{2}{7}\alpha_2 + \tfrac{1}{7}\alpha_1\right)(\vec{v}+3\vec{w})$$

$$= \left[\tfrac{6}{7}\alpha_1 - \cancel{\tfrac{2}{7}\alpha_2} + \cancel{\tfrac{2}{7}\alpha_2} + \tfrac{1}{7}\alpha_1\right]\vec{v} + \left[-\cancel{\tfrac{3}{7}\alpha_1} + \tfrac{1}{7}\alpha_2 + \tfrac{6}{7}\alpha_2 + \cancel{\tfrac{3}{7}\alpha_1}\right]\vec{w}$$

$$= \alpha_1\vec{v} + \alpha_2\vec{w} = \vec{y} \checkmark \quad \blacksquare$$

(* PHEW! *)

8. $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$; $\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

(a) $\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} \in$ SPAN$(\mathcal{C})$ IF $\exists\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ WITH $\alpha_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \alpha_3\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$:

$$\begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & 0 & 1 & -3 \\ 3 & 1 & 2 & -1 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15/4 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & 9/2 \end{array}\right] \text{ CONSISTENT } \checkmark$$

(SPECIFICALLY, $\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} = -\frac{15}{4}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{9}{2}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$)

$\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} \in$ SPAN$(\mathcal{D})$ IF $\exists\, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ WITH $\beta_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \beta_2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \beta_3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$:

$$\begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & -1 & -1 & -3 \\ -1 & 1 & 0 & -1 \end{array}\right] \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{cccc} & & \text{FREE} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ CONSISTENT } \checkmark$$

(SINCE THERE IS A FREE VARIABLE, WE CAN WRITE THIS VECTOR AS A L.C. IN INFINITELY MANY WAYS!)

(b) $[\mathcal{C} \mid \mathcal{D}] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & -1 & -1 \\ 3 & 1 & 2 & -1 & 1 & 0 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5/4 & -5/4 \\ 0 & 1 & 0 & -1 & \frac{7}{4} & \frac{3}{4} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & \frac{3}{2} \end{array}\right]$$

CONSISTENT FOR ALL THREE COLUMNS OF THE AUGMENTATION ✓

(IF WE WERE PARTICULARLY OBSERVANT OF PART (a)'S SOLUTION, WE'D NOTE THAT SINCE EVERY ROW IS A PIVOT ROW FOR $[\mathcal{C} \mid \dots]$, NO MATTER WHAT WE FILL IN, WE'LL GET CONSISTENT SYSTEMS!)

(c) $[\mathcal{D} \mid \mathcal{C}] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & 2 & 0 & 1 \\ -1 & 1 & 0 & 3 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 6 & 0 & 5 \end{array}\right] \leftarrow !$

INCONSISTENT FOR THE FIRST + THIRD COLUMNS OF THE AUGMENTATION, SO NEITHER OF THE CORRESPONDING VECTORS, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ NOR $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, ARE IN SPAN$(\mathcal{D})$!

(d) THE COLLECTION $\mathcal{D}$ CAN'T POSSIBLY SPAN $\mathbb{R}^3$ — SPAN$(\mathcal{D}) \neq \mathbb{R}^3$ BECAUSE, IN PARTICULAR, EITHER OF THE TWO VECTORS FOUND IN PART (c) ARE IN $\mathbb{R}^3$ BUT NOT SPAN$(\mathcal{D})$.

(e) $\mathcal{C}$ DOES SPAN $\mathbb{R}^3$:

$$[\mathcal{C}] \rightsquigarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PIVOT IN EVERY ROW — $\therefore$ THIS WOULD BE CONSISTENT FOR ANY AUGMENTATION; I.E., EVERY $\vec{v} \in \mathbb{R}^3$ IS IN THE SPAN OF $\mathcal{C}$, SO $\mathcal{C}$ SPANS $\mathbb{R}^3$.

(AS IN PART (b), WE COULD HAVE NOTED FROM OUR WORK IN PART (a) THAT THIS SYSTEM WAS GUARANTEED TO HAVE A PIVOT IN EVERY ROW, AND THUS TO BE CONSISTENT, REGARDLESS OF WHAT WE CHOOSE TO AUGMENT $[\mathcal{C} \mid \dots]$ WITH!)

(f) TO SHOW THAT $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ IS A L.C. OF $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$,

JUST SHOW THAT THE SYSTEM $\alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ IS CONSISTENT:

$$\begin{array}{c} \begin{array}{cc} \alpha_1 & \alpha_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \text{CONSISTENT} \checkmark$$

(SPECIFICALLY, $\alpha_1 = 1$ AND $\alpha_2 = 1$, SO $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$)

(g) TO SHOW THAT $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ IS NOT A L.C. OF $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$,

JUST SHOW THAT THE SYSTEM $\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ IS INCONSISTENT:

$$\begin{array}{c} \begin{array}{cc} \alpha_1 & \alpha_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 2 \end{array}\right] \to \text{INCONSISTENT!} \checkmark$$

(h) TO FIND A CONDITION ON a, b, AND c THAT DETERMINES WHETHER OR NOT THE VECTOR $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ IS IN SPAN$(\mathcal{D})$, SET UP THE SYSTEM $\left[\mathcal{D} \,\middle|\, \begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right]$ AND REDUCE:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -1 & -1 & b \\ -1 & 1 & 0 & c \end{array}\right] \xrightarrow{+1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & -1 & -1 & b \\ 0 & 1 & 1 & a+c \end{array}\right] \cdot (-1)$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -b \\ 0 & 1 & 1 & a+c \end{array}\right] \xrightarrow{-1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & -b \\ 0 & 0 & 0 & a+b+c \end{array}\right] \leftarrow !$$

THIS SYSTEM WILL BE CONSISTENT IF, AND ONLY IF, $\underline{a+b+c=0}$

(IF $a+b+c=0$, WE GET $[0 \; 0 \; 0 \mid 0]$, WHICH IS FINE; ON THE OTHER HAND, IF $a+b+c \neq 0$, WE GET $[0 \; 0 \; 0 \mid *]$, WHICH GIVES AN INCONSISTENT SYSTEM!)