$V$: N-DIMENSIONAL VECTOR SPACE; $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ A BASIS FOR $V$.

1. A MULTILINEAR FORM ON $V$ IS A FUNCTION THAT ASSIGNS TO EACH ORDERED COLLECTION OF K VECTORS IN $V$ SOME SCALAR AND SATISFIES THE PROPERTY OF MULTILINEARITY (LINEARITY IN EACH OF ITS ARGUMENTS INDIVIDUALLY).

I.E., $\Phi$ IS A K-LINEAR FORM IF $\Phi: V^k \to \mathbb{R}$ SATISFIES

① $\Phi(\vec{v}_1 + \vec{v}_1{}', \vec{v}_2, \ldots, \vec{v}_k) = \Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) + \Phi(\vec{v}_1{}', \vec{v}_2, \ldots, \vec{v}_k)$

$\Phi(a\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = a\,\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

② $\Phi(\vec{v}_1, \vec{v}_2 + \vec{v}_2{}', \ldots, \vec{v}_k) = \Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) + \Phi(\vec{v}_1, \vec{v}_2{}', \ldots, \vec{v}_k)$

$\Phi(\vec{v}_1, a\vec{v}_2, \ldots, \vec{v}_k) = a\,\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

⋮

Ⓚ $\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k + \vec{v}_k{}') = \Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) + \Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k{}')$

$\Phi(\vec{v}_1, \vec{v}_2, \ldots, a\vec{v}_k) = a\,\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

2. USING MULTILINEARITY AND A BASIS $\mathcal{B} = (\vec{b}_1, \ldots, \vec{b}_n)$ FOR $V$, WE CAN SPLIT A K-FORM'S VALUE INTO "ATOMIC" PIECES TO SEE HOW MANY FREE CHOICES WE HAVE WHEN CONSTRUCTING ONE:

- 1-FORM: $\Phi: V \to \mathbb{R}$ 'S VALUES CAN BE SPLIT AS:

$$\Phi(a_1\vec{b}_1 + a_2\vec{b}_2 + \ldots + a_n\vec{b}_n)$$
$$= a_1\Phi(\vec{b}_1) + a_2\Phi(\vec{b}_2) + \cdots + a_n\Phi(\vec{b}_n) \quad \text{BY LINEARITY,}$$

SO WE CAN FREELY CHOOSE THE $n$ VALUES $\Phi(\vec{b}_1), \ldots, \Phi(\vec{b}_n)$ TO OBTAIN A 1-FORM ON $V$.

- 2-FORM: $\Phi: V^2 \to \mathbb{R}$'S VALUES CAN BE SPLIT AS:

$$\Phi(a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n,\ \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \cdots + \beta_n\vec{b}_n)$$

$$\begin{aligned}
&= a_1\Phi(\vec{b}_1, \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \cdots + \beta_n\vec{b}_n) \\
&+ a_2\Phi(\vec{b}_2, \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \cdots + \beta_n\vec{b}_n) \\
&\ \ \vdots \\
&+ a_n\Phi(\vec{b}_n, \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \cdots + \beta_n\vec{b}_n)
\end{aligned}$$

(BY LINEARITY IN THE FIRST ARGUMENT)

$$\begin{aligned}
&= a_1\left[\beta_1\Phi(\vec{b}_1, \vec{b}_1) + \beta_2\Phi(\vec{b}_1, \vec{b}_2) + \cdots + \beta_n\Phi(\vec{b}_1, \vec{b}_n)\right] \\
&+ a_2\left[\beta_1\Phi(\vec{b}_2, \vec{b}_1) + \beta_2\Phi(\vec{b}_2, \vec{b}_2) + \cdots + \beta_n\Phi(\vec{b}_2, \vec{b}_n)\right] \\
&\ \ \vdots \\
&+ a_n\left[\beta_1\Phi(\vec{b}_n, \vec{b}_1) + \beta_2\Phi(\vec{b}_n, \vec{b}_2) + \cdots + \beta_n\Phi(\vec{b}_n, \vec{b}_n)\right]
\end{aligned}$$

(BY LINEARITY IN THE SECOND ARGUMENT)

$$\begin{aligned}
&= a_1\beta_1\Phi(\vec{b}_1, \vec{b}_1) + a_1\beta_2\Phi(\vec{b}_1, \vec{b}_2) + \cdots + a_1\beta_n\Phi(\vec{b}_1, \vec{b}_n) \\
&+ a_2\beta_1\Phi(\vec{b}_2, \vec{b}_1) + a_2\beta_2\Phi(\vec{b}_2, \vec{b}_2) + \cdots + a_2\beta_n\Phi(\vec{b}_2, \vec{b}_n) \\
&\ \ \vdots \\
&+ a_n\beta_1\Phi(\vec{b}_n, \vec{b}_1) + a_n\beta_2\Phi(\vec{b}_n, \vec{b}_2) + \cdots + a_n\beta_n\Phi(\vec{b}_n, \vec{b}_n),
\end{aligned}$$

SO WE CAN FREELY CHOOSE $n^2$ ($= n \cdot n$ — $n$ CHOICES FOR THE FIRST ARGUMENT AND $n$ FOR THE SECOND) VALUES $\Phi(\vec{b}_1, \vec{b}_1), \ldots, \Phi(\vec{b}_n, \vec{b}_n)$ TO OBTAIN A 2-FORM ON $V$.

- n-FORM: $\Phi: V^n \to \mathbb{R}$'S VALUES CAN BE OBTAINED SIMILARLY BY SPLITTING INTO $\Phi$ OPERATING ON BASIS VECTORS IN EACH ARGUMENT, SO WE CAN FREELY CHOOSE $n^n$ ($= n \cdot n \cdots n$ — $n$ CHOICES FOR EACH ARGUMENT) VALUES $\Phi(\vec{b}_1, \ldots, \vec{b}_1), \ldots, \Phi(\vec{b}_n, \ldots, \vec{b}_n)$ TO OBTAIN AN n-FORM ON $V$.

3. AN ALTERNATING FORM ON $V$ IS A MULTILINEAR FORM $\Phi$ THAT SATISFIES:

- $\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = 0$ IF ANY TWO OF ITS ARGUMENTS ARE EQUAL.

CONSEQUENCE: SWITCHING TWO ARGUMENTS OF AN ALTERNATING FORM $\Phi$ NEGATES ITS VALUE.

PROOF: WLOG., SUPPOSE THAT WE WANT TO SWITCH $\vec{v}_1 + \vec{v}_2$ IN $\Phi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$.

WELL, $0 = \Phi(\vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)$ BECAUSE $\Phi$ IS ALTERNATING

$$= \Phi(\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n) + \Phi(\vec{v}_2, \vec{v}_1 + \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)$$

$$= \underbrace{\Phi(\vec{v}_1, \vec{v}_1, \vec{v}_3, \ldots, \vec{v}_n)}_{\text{ZERO!}} + \Phi(\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)$$

$$+ \Phi(\vec{v}_2, \vec{v}_1, \vec{v}_3, \ldots, \vec{v}_n) + \underbrace{\Phi(\vec{v}_2, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)}_{\text{ZERO!}}$$

$$= \Phi(\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n) + \Phi(\vec{v}_2, \vec{v}_1, \vec{v}_3, \ldots, \vec{v}_n),$$

SO SOLVING, $\Phi(\vec{v}_2, \vec{v}_1, \vec{v}_3, \ldots, \vec{v}_n) = -\Phi(\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n)$ ■

THIS IS WHY THE ADJECTIVE "ALTERNATING" IS USED

ADJUSTING OUR RESULTS FROM #2,

- 1-FORM: BECAUSE A 1-FORM ONLY HAS ONE ARGUMENT, EVERY 1-FORM IS ALTERNATING, BY DEFAULT.
  $\therefore$ WE AGAIN HAVE $n$ FREE CHOICES WHEN CONSTRUCTING AN ALTERNATING 1-FORM.

- 2-FORM: AN ALTERNATING 2-FORM MUST HAVE
  - $\Phi(\vec{b}_i, \vec{b}_i) = 0$ — KILLING $n$ OF OUR FREE CHOICES
  - $\Phi(\vec{b}_i, \vec{b}_j) = -\Phi(\vec{b}_j, \vec{b}_i)$ — CUTTING THE REST IN HALF

  $\therefore$ WE HAVE $\frac{1}{2}(n^2 - n)$ FREE CHOICES WHEN CONSTRUCTING AN ALTERNATING 2-FORM.

- $n$-FORM: OF THE $n^n$ FREE CHOICES FOR AN $n$-FORM, $\Phi$ MUST MAP ANY INPUT WITH A REPEATED ARGUMENT TO 0; BUT ANY INPUT THAT DOESN'T REPEAT A BASIS VECTOR $\vec{b}_i$ MUST USE EACH ONE EXACTLY ONCE — BY ITERATIVELY SWITCHING ARGUMENTS, THESE OTHER VALUES OF $\Phi(\vec{b}_{\ldots}, \ldots, \vec{b}_{\ldots})$ MUST EQUAL $\pm\Phi(\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$

  $\therefore$ WE HAVE ONLY ONE FREE CHOICE WHEN CONSTRUCTING AN ALTERNATING $n$-FORM ON $V$!

4. $V$: VECTOR SPACE WITH BASIS $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$;
$Q$: ALTERNATING 2-FORM ON $V$.

$$Q(a_{11}\vec{b}_1 + a_{21}\vec{b}_2, a_{12}\vec{b}_1 + a_{22}\vec{b}_2) = \ldots \quad \text{(BILINEARITY)}$$

$$= a_{11}a_{12}\cancel{Q(\vec{b}_1, \vec{b}_1)}^{0} + a_{11}a_{22}Q(\vec{b}_1, \vec{b}_2) + a_{21}a_{12}Q(\vec{b}_2, \vec{b}_1) + a_{21}a_{22}\cancel{Q(\vec{b}_2, \vec{b}_2)}^{0}$$

(ALTERNATING)

$$= a_{11}a_{22}Q(\vec{b}_1, \vec{b}_2) + a_{21}a_{12}(-Q(\vec{b}_1, \vec{b}_2))$$

$$= (a_{11}a_{22} - a_{21}a_{12})\,Q(\vec{b}_1, \vec{b}_2).$$

IF WE SWITCH $a_{ij} \leftrightarrow a_{ji}$, THIS BECOMES $(a_{11}a_{22} - a_{12}a_{21})\,Q(\vec{b}_1, \vec{b}_2)$, WHICH EQUALS THE SAME THING!

5. $V$: VECTOR SPACE WITH BASIS $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$;
$T$: ALTERNATING 3-FORM ON $V$.

$$T(a_{11}\vec{b}_1 + a_{21}\vec{b}_2 + a_{31}\vec{b}_3,\ a_{12}\vec{b}_1 + a_{22}\vec{b}_2 + a_{32}\vec{b}_3,\ a_{13}\vec{b}_1 + a_{23}\vec{b}_2 + a_{33}\vec{b}_3)$$

$= \ldots$ (TRILINEARITY) — NOTE THAT ANY OF THE 27 TERMS WITH REPEATED $\vec{b}$'S EVAPORATE; ONLY THE FOLLOWING SIX SURVIVE:

$$= a_{11}a_{22}a_{33}T(\vec{b}_1, \vec{b}_2, \vec{b}_3) + a_{11}a_{32}a_{23}T(\vec{b}_1, \vec{b}_3, \vec{b}_2)$$

$$+ a_{21}a_{12}a_{33}T(\vec{b}_2, \vec{b}_1, \vec{b}_3) + a_{21}a_{32}a_{13}T(\vec{b}_2, \vec{b}_3, \vec{b}_1)$$

$$+ a_{31}a_{12}a_{23}T(\vec{b}_3, \vec{b}_1, \vec{b}_2) + a_{31}a_{22}a_{13}T(\vec{b}_3, \vec{b}_2, \vec{b}_1)$$

$$= (a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13})\,T(\vec{b}_1, \vec{b}_2, \vec{b}_3).$$

IF WE SWITCH $a_{ij} \leftrightarrow a_{ji}$, THIS AGAIN STAYS THE SAME.

6. DET : $\mathbb{R}^{n\times n} \to \mathbb{R}$ IS THE UNIQUE ALTERNATING ①, MULTILINEAR ② n-FORM FOR WHICH $\det(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) = 1$ ③. → I.E., $\det I_n = 1$

THESE THREE PROPERTIES CHARACTERIZE THE FUNCTION DET BECAUSE THE ONLY FREE CHOICE IN DEFINING AN ALTERNATING n-FORM ON THE n-DIMENSIONAL VECTOR SPACE $\mathbb{R}^n$ IS ITS VALUE ON ANY ONE BASIS (SEE #3) — PROPERTY ③ DETERMINES THIS, SO THERE IS EXACTLY ONE SUCH FUNCTION THAT HAS ALL THREE PROPERTIES.

7. THE TRANSPOSE, $A^T$, OF AN $n\times n$ MATRIX $A$ IS THE $n\times n$ MATRIX OBTAINED BY SWITCHING THE $(i,j)^{TH}$ ENTRY (ROW, COL) WITH THE $(j,i)^{TH}$ ENTRY — I.E., FLIPPING A ACROSS ITS DIAGONAL.

(E.G., IF $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, THEN $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.)

* $\det A^T = \det A$, AS OBSERVED IN #4 + #5

(THE THREE PROPERTIES THAT CHARACTERIZE DET FORCE IT TO HAVE THE SAME VALUE FOR $A^T$ AS FOR $A$.)

8. "EXPANSION BY MINORS" ALLOWS US TO COMPUTE THE DETERMINANT OF A LARGE MATRIX IN TERMS OF THE DETERMINANTS OF [SLIGHTLY!] SMALLER ONES:

- FIGURATIVELY MARK THE ENTRIES WITH + OR −, STARTING WITH + AT THE TOP-LEFT AND ALTERNATING AS YOU MOVE DOWNWARD OR RIGHTWARD.
- CHOOSE A ROW (OR COLUMN) TO EXPAND UPON, AND FOR EACH OF ITS ENTRIES, ADD OR SUBTRACT (AS MARKED ABOVE) THE ENTRY TIMES THE DETERMINANT OF ITS MINOR IN A. — THE RESULT EQUALS DET A.

↳ THE MATRIX OBTAINED BY CROSSING OUT THAT ENTRY'S ENTIRE ROW + COLUMN.

E.G., EXPANDING $\det \begin{bmatrix} {}^{+}1 & 5 & 7 \\ {}^{-}0 & 4 & 3 \\ {}^{+}-1 & 2 & 6 \end{bmatrix}$ ON THE FIRST COLUMN:

$$= +(1)\det\begin{bmatrix} 4 & 3 \\ 2 & 6 \end{bmatrix} - (0)\begin{bmatrix} 5 & 7 \\ 2 & 6 \end{bmatrix} + (-1)\begin{bmatrix} 5 & 7 \\ 4 & 3 \end{bmatrix}$$

$$= \ldots$$

THIS METHOD CAN BE JUSTIFIED BY USING THE MULTILINEARITY OF DET ON THE CHOSEN ROW OR COLUMN AND CAREFULLY ANALYZING THE RESULTING SIMPLER PIECES.

9. BELOW ARE EXPLAINED THE EFFECTS OF BASIC COLUMN OPERATIONS ON THE DETERMINANT; NOTE THAT THE SAME RESULTS HOLD FOR ROW OPERATIONS, WHICH CAN BE ACHIEVED AS FOLLOWS:

① TRANSPOSE THE MATRIX
② PERFORM THE CORRESPONDING COLUMN OPERATION
③ TRANSPOSE THE MATRIX AGAIN.

BECAUSE ① + ③ DON'T AFFECT THE DETERMINANT, WE THUS OBTAIN THE SAME RULES FOR ROW OPERATIONS AS WE DO FOR COLUMN OPERATIONS.

(a) INTERCHANGING TWO COLUMNS OF A NEGATES THE DETERMINANT, WHICH IS A CONSEQUENCE OF DET BEING ALTERNATING (SEE #3)

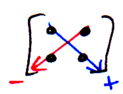
(b) SCALING A COLUMN BY $q$ SCALES THE DETERMINANT BY $q$, BECAUSE DET IS MULTILINEAR.

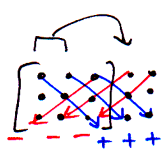
(c) ADDING A MULTIPLE OF ONE COLUMN TO ANOTHER DOESN'T AFFECT THE DETERMINANT, BECAUSE DET IS BILINEAR AND ALTERNATING.

$$\det(\vec{v}_1 + q\vec{v}_k, \vec{v}_2, \ldots, \vec{v}_n)$$
$$= \det(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) + \cancel{q\det(\vec{v}_k, \vec{v}_2, \ldots, \vec{v}_n)} \rightarrow 0 : \vec{v}_k \text{ REPEATED!}$$

10. $\det[a] = a \cdot \det[1] = a \cdot \det I_1 = a \cdot 1 = a$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$




$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3$$

* THESE SHORTCUTS ABSOLUTELY, POSITIVELY, DO NOT WORK FOR LARGER MATRICES!

11. THE INVERSE OF AN INVERTIBLE MATRIX A CAN BE COMPUTED VIA DETERMINANTS AS FOLLOWS:

- FORM A MATRIX WHOSE $(i,j)^{\text{TH}}$ ENTRY IS THE DETERMINANT OF THE $(i,j)^{\text{TH}}$ MINOR OF A
- CHANGE THE SIGNS OF THIS MATRIX ACCORDING TO THE USUAL RULE (AS WHEN COMPUTING DETERMINANTS VIA MINORS).
- TRANSPOSE.
- SCALE BY $\frac{1}{\det A}$

— THE RESULTING MATRIX GIVES $A^{-1}$.

WHY DOES THIS WORK? WITH $A^{-1}$ AS COMPUTED ABOVE, IF WE COMPUTE $AA^{-1}$ (OR $A^{-1}A$), WE FIND, MINOR EXPANSIONS FOR DET A ALONG THE DIAGONAL; THE $\frac{1}{\det A}$ MAKES THE DIAGONAL ENTRIES 1. OFF THE DIAGONAL, WE FIND MINOR EXPANSIONS FOR DETERMINANTS OF MATRICES HAVING DUPLICATE ROWS (OR COLUMNS), GIVING ZEROS.

12. (a)
$$\det\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \lambda_1 \det\begin{bmatrix} 1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$
$$= \lambda_1\lambda_2 \det\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$
$$= \cdots = \lambda_1\lambda_2\cdots\lambda_n \det\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \quad \text{(BY MULTILINEARITY)}$$
$$= \lambda_1\lambda_2\cdots\lambda_n \det I_n$$
$$= \lambda_1\lambda_2\cdots\lambda_n \checkmark \quad (\det I_n = 1)$$

(b) $\det(AB) = (\det A)(\det B)$: (OUTLINE OF ARGUMENT)

WE CAN EASILY SHOW THAT THE FUNCTION $\Phi: X \longmapsto \det(AX)$ IS MULTILINEAR AND ALTERNATING; THUS, FROM #3, THE FUNCTION $\Phi$ MUST BE SOME SCALAR MULTIPLE OF DET.

SAY $\Phi = q \det$; THEN $\Phi(I) = q \cdot \det(I) = q \cdot 1 = q$.

BUT ON THE OTHER HAND, $\Phi(I) \overset{\text{DEF}}{=} \det(AI) = \det A$, SO $q = \det A$.

CONCLUSION: $\det(AX) = \Phi(X) = q \det(X) = (\det A)(\det X)$,

SO SETTING $X = B$, WE HAVE $\det(AB) = (\det A)(\det B)$ ✓

THIS IS MULTIPLICATION OF SCALARS, SO IT'S COMMUTATIVE!

(c) IF $A$ IS INVERTIBLE, THEN $\det(A^{-1}) = (\det A)^{-1}$:

BY DEFINITION OF INVERSE, $A^{-1}A = I$,

SO $\det(A^{-1}A) = \det I = 1$

BUT BY (b), $\det(A^{-1}A) = (\det A^{-1})(\det A)$,

SO $(\det A^{-1})(\det A) = 1$,

THUS, $\det A^{-1}$ IS THE MULTIPLICATIVE INVERSE OF $\det A$,

I.E., $\det(A^{-1}) = (\det A)^{-1}$ ✓

* NOTE THAT THIS MEANS THE DETERMINANT OF AN INVERTIBLE MATRIX MUST BE NONZERO!

(d) $\det(qA) = q^n \det A$: $qA$ IS $A$ SCALED BY $q$ IN EACH COLUMN.

THUS, WE CAN APPLY MULTILINEARITY ONE COLUMN AT A TIME TO FACTOR OUT $n$ $q$'s, GIVING $\det(qA) = q^n \det A$. ✓

(e) IF $A \sim B$, THEN $\det A = \det B$:

SUPPOSE THAT $A \sim B$. THEN $\exists$ INVERTIBLE $X$ WITH $B = XAX^{-1}$.

BUT THEN $\det B = \det(XAX^{-1}) = (\det X)(\det A)(\det X^{-1})$

$$= \underbrace{(\det X)(\det X^{-1})}_{1\text{, FROM PART (c)}}(\det A)$$

$= \det A$. ✓