$L: V \to V$ ENDOMORPHISM; V: n-DIMENSIONAL VECTOR SPACE.

1. GIVEN AN ENDOMORPHISM $L: V \to V$, A NONZERO VECTOR $\vec{v} \in V$ IS AN EIGENVECTOR OF L WITH EIGENVALUE λ IF $L\vec{v} = \lambda\vec{v}$.
   ($\vec{v}$ AND λ COME AS A PAIR) SCALAR

   ... SO:
   - A NONZERO VECTOR $\vec{v} \in V$ IS AN EIGENVECTOR OF L IF $\exists$ SOME SCALAR λ SUCH THAT $L\vec{v} = \lambda\vec{v}$.
   - A SCALAR λ IS AN EIGENVALUE OF L IF $\exists$ SOME NONZERO VECTOR $\vec{v} \in V$ SUCH THAT $L\vec{v} = \lambda\vec{v}$

2. AS WE OBSERVED IN THE PREVIOUS PROBLEM SET,

   IF $[L]_{\mathcal{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ FOR SOME BASIS $\mathcal{V} = (\vec{v}_1, \ldots, \vec{v}_n)$ OF V,

   THEN EACH $\vec{v}_i$ IS AN EIGENVECTOR OF L WITH EIGENVALUE λ_i. CONVERSELY, IF $\mathcal{V} = (\vec{v}_1, \ldots, \vec{v}_n)$ IS A BASIS OF V CONSISTING OF EIGENVECTORS OF L, WITH CORRESPONDING EIGENVECTORS $\lambda_1, \ldots, \lambda_n$, THEN $[L]_{\mathcal{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ (DIRECT COMPUTATION).

   * IN SUM: $L: V \to V$ IS DIAGONALIZABLE $\Leftrightarrow$ V HAS AN EIGENBASIS.
   (I.E., A BASIS CONSISTING OF EIGENVECTORS OF L)

3. CLAIM: IF $\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ CONSISTS OF EIGENVECTORS OF L WITH DISTINCT EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_k$, THEN $\mathcal{C}$ IS LINEARLY INDEPENDENT.

   PROOF: SUPPOSE THAT $\mathcal{C} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, WHERE EACH $\vec{v}_i$ IS AN EIGENVECTOR OF L WITH EIGENVALUE λ_i, AND THAT THE EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_k$ ARE DISTINCT.

   GOAL: SHOW THAT $\mathcal{C}$ IS LINEARLY INDEPENDENT, I.E., $\nexists$ NONTRIVIAL LINEAR RELATION ON $\mathcal{C}$

   * (THIS IS AN EQUIVALENT FORMULATION OF LINEAR INDEPENDENCE, WHICH WILL BE THE EASIEST ONE TO USE HERE, WITH A LITTLE TRICK... WE'LL PROVE THIS BY CONTRADICTION, SUPPOSING THAT $\exists$ A NONTRIVIAL RELATION ON $\mathcal{C}$ AND SHOWING THIS IS IMPOSSIBLE)

   SUPPOSE THAT $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$ IS A NONTRIVIAL LINEAR RELATION ON $\mathcal{C}$ WITH AS FEW NONZERO COEFFICIENTS AS POSSIBLE.
   THIS IS THE TRICK!

   BECAUSE IT IS A NONTRIVIAL LINEAR RELATION, AT LEAST ONE COEFFICIENT IS NONZERO — W.L.O.G., $a_1 \neq 0$.

   NOW, $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}$, SO APPLYING L,

   $$\begin{aligned} \vec{0} &= L(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) \\ &= a_1 L\vec{v}_1 + a_2 L\vec{v}_2 + \cdots + a_k L\vec{v}_k \quad \text{BY LINEARITY OF } L \\ &= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 + \cdots + a_k\lambda_k\vec{v}_k \quad (\vec{v}_i: \text{EIGENVECTORS}) \end{aligned}$$

   BUT SCALING OUR ORIGINAL LINEAR RELATION BY λ_1 GIVES US:

   $$a_1\lambda_1\vec{v}_1 + a_2\lambda_1\vec{v}_2 + \cdots + a_k\lambda_1\vec{v}_k = \vec{0}$$

   SO SUBTRACTING THESE LAST TWO EQUATIONS,

   $$\underbrace{a_1(\lambda_1-\lambda_1)}_{0}\vec{v}_1 + a_2(\lambda_2-\lambda_1)\vec{v}_2 + \cdots + a_k(\lambda_k-\lambda_1)\vec{v}_k = \vec{0}$$

THIS GIVES US A NEW LINEAR RELATION ON $\mathcal{E}$; IF ANY OF THE ORIGINAL COEFFICIENTS $a_2, \ldots, a_k$ WERE NONZERO, WE'D HAVE A LINEAR RELATION ON $\mathcal{E}$ WITH ONE FEWER NONZERO COEFFICIENT — SO SINCE OUR ORIGINAL L.R. HAD AS FEW AS POSSIBLE, IT MUST BE THE CASE THAT $a_2, \ldots, a_k = 0$. BUT THEN OUR ORIGINAL RELATION BECOMES $a_1\vec{v}_1 = \vec{0}$, SO SINCE $a_1 \neq 0$, $\frac{1}{a_1}a_1\vec{v}_1 = \frac{1}{a_1}\vec{0}$, I.E., $\vec{v}_1 = \vec{0}$, WHICH IS IMPOSSIBLE BECAUSE BY HYPOTHESIS, $\vec{v}_1$ IS AN EIGENVECTOR OF L.

IN CONCLUSION, NO NONTRIVIAL L.R. ON $\mathcal{E}$ EXISTS, SO BY DEFINITION, $\mathcal{E}$ IS LINEARLY INDEPENDENT. ■

* COROLLARY: IF L HAS n DISTINCT EIGENVALUES $\lambda_1, \ldots, \lambda_n$, THEN FOR EACH λ_i $\exists$ SOME EIGENVECTOR $\vec{v}_i$. BY THE PREVIOUS RESULT, $\mathcal{V} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ IS LINEARLY INDEPENDENT IN THE n-DIMENSIONAL VECTOR SPACE V, AND THUS IS A BASIS FOR V, I.E., $\mathcal{V}$ IS AN EIGENBASIS FOR V, SO L IS DIAGONALIZABLE!

4. SUPPOSE THAT A IS A SQUARE MATRIX.

(a) CLAIM: $\vec{x} \neq \vec{0}$ IS AN EIGENVECTOR OF A WITH EIGENVALUE λ $\Leftrightarrow$ $\vec{x} \in \ker(A - \lambda I)$.

PROOF: $\vec{x} \neq \vec{0}$ IS AN EIGENVECTOR OF A WITH EIGENVALUE λ

$\Leftrightarrow A\vec{x} = \lambda\vec{x}$

$\Leftrightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$

$\Leftrightarrow A\vec{x} - \lambda I\vec{x} = \vec{0}$

$\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}$

$\Leftrightarrow \vec{x} \in \ker(A - \lambda I)$ ■

(b) WE CAN THUS DETERMINE THE EIGENVECTORS OF A FOR ANY GIVEN EIGENVALUE λ BY FINDING $\ker(A - \lambda I)$, I.E., BY SOLVING THE HOMOGENEOUS SYSTEM $(A - \lambda I)\vec{x} = \vec{0}$.

COMPUTATIONALLY: SET UP AND SOLVE $[A - \lambda I]$

5. AN UPPER-TRIANGULAR MATRIX IS A SQUARE MATRIX WHOSE ONLY NONZERO ENTRIES LIE ON OR ABOVE THE DIAGONAL. (SAID ANOTHER WAY, ALL ENTRIES BELOW THE DIAGONAL ARE ZERO.)

* THE EIGENVALUES OF AN UPPER-TRIANGULAR MATRIX ARE ITS DIAGONAL ENTRIES:

λ IS AN EIGENVALUE OF A

$\Leftrightarrow \ker(A - \lambda I) \neq \{\vec{0}\}$ BY THE PREVIOUS PROBLEM

$\Leftrightarrow A - \lambda I$ DOESN'T GIVE A PIVOT IN EVERY COLUMN WHEN REDUCED

BUT IF $A = \begin{bmatrix} a_1 & & * \\ & a_2 & \\ & & \ddots \\ 0 & & & a_n \end{bmatrix}$, THEN $A - \lambda I = \begin{bmatrix} a_1 - \lambda & & * \\ & a_2 - \lambda & \\ & & \ddots \\ 0 & & & a_n - \lambda \end{bmatrix}$

IF λ MATCHES A DIAGONAL ENTRY OF A, THEN THAT ENTRY WILL BECOME ZERO IN $A - \lambda I$, AND WE'LL BE MISSING AT LEAST ONE PIVOT WHEN WE REDUCE, SO AS ABOVE, λ WILL BE AN EIGENVALUE OF A.

CONVERSELY, IF λ MATCHES NONE OF THE DIAGONAL ENTRIES OF A, THEN THE DIAGONAL ENTRIES OF $A - \lambda I$ WILL ALL BE NONZERO, SO WE CAN "KILL UPWARD" FROM THEM TO REDUCE $A - \lambda I$ TO A MATRIX WITH A PIVOT IN EVERY COLUMN — SO λ WILL NOT BE AN EIGENVALUE OF A.

6. THE ONLY EIGENVALUE OF A IS 1, BECAUSE A IS AN UPPER-TRIANGULAR MATRIX WHOSE DIAGONAL ENTRIES ARE BOTH 1'S.

TO DETERMINE THE CORRESPONDING EIGENVECTORS, WE JUST FIND THE KERNEL OF $A-1I$, I.E., WE SOLVE $(A-I)\vec{x}=\vec{0}$:

$$\begin{bmatrix} 1-1 & 5 \\ 0 & 1-1 \end{bmatrix} = \begin{array}{cc} x_1 & x_2 \\ \end{array}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

FREE

x_1: FREE

$x_2 = 0$

$$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\ker(A-I)$ THUS HAS BASIS $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, SO THE MOST EIGENVECTORS WE CAN HAVE IN A BASIS FOR $\mathbb{R}^2$ IS ONE (OTHERWISE, WE'D HAVE TWO VECTORS FROM THIS ONE-DIMENSIONAL SUBSPACE, WHICH WOULD RESULT IN LINEAR DEPENDENCE).

$\hookrightarrow \vec{0}$ AND THE EIGENVECTORS WITH EIGENVALUE 1

$\therefore$ THERE CAN BE NO EIGENBASIS FOR $\mathbb{R}^2$, AND THUS A IS NOT DIAGONALIZABLE.

7. $L: V \to V$ ENDOMORPHISM; $E_\lambda = \{\vec{v} \in V : L\vec{v} = \lambda\vec{v}\}$ FOR EACH $\lambda \in \mathbb{R}$.

(a) E_λ IS A SUBSPACE OF V:

[⓪ $\vec{0} \in E_\lambda$] $\vec{0} \in V$ AND $L\vec{0} = \vec{0} = \lambda\vec{0}$, SO BY DEFINITION OF E_λ, $\vec{0} \in E_\lambda$. ✓

[① $\forall \vec{v}_1 \in E_\lambda$ AND SCALAR α, $\alpha\vec{v}_1 \in E_\lambda$]

LET $\vec{v}_1 \in E_\lambda$ AND SCALAR α BE GIVEN;

BY DEFINITION OF E_λ, $L\vec{v}_1 = \lambda\vec{v}_1$. $(*)$

WELL, $\alpha\vec{v}_1 \in V$ AND
$$\begin{aligned} L(\alpha\vec{v}_1) &= \alpha L\vec{v}_1 && \text{BY LINEARITY} \\ &= \alpha(\lambda\vec{v}_1) && \text{FROM } (*) \\ &= \lambda(\alpha\vec{v}_1), \end{aligned}$$

SO BY DEFINITION OF E_λ, $\alpha\vec{v}_1 \in E_\lambda$. ✓

[② $\forall \vec{v}_1, \vec{v}_2 \in E_\lambda$, $\vec{v}_1 + \vec{v}_2 \in E_\lambda$]

LET $\vec{v}_1, \vec{v}_2 \in E_\lambda$ BE GIVEN.

BY DEFINITION OF E_λ, $L\vec{v}_1 = \lambda\vec{v}_1$ AND $L\vec{v}_2 = \lambda\vec{v}_2$. $(*)$

WELL, $\vec{v}_1 + \vec{v}_2 \in V$ AND
$$\begin{aligned} L(\vec{v}_1+\vec{v}_2) &= L\vec{v}_1 + L\vec{v}_2 && \text{BY LINEARITY} \\ &= \lambda\vec{v}_1 + \lambda\vec{v}_2 && \text{FROM } (*) \\ &= \lambda(\vec{v}_1+\vec{v}_2), \end{aligned}$$

SO BY DEFINITION OF E_λ, $\vec{v}_1 + \vec{v}_2 \in E_\lambda$. ✓ ∎

(b) $\dim E_\lambda > 0 \Rightarrow E_\lambda \neq \{\vec{0}\}$, SO $\exists$ SOME NONZERO VECTOR $\vec{v} \in E_\lambda$. BUT THEN THIS VECTOR HAS $L\vec{v} = \lambda\vec{v}$, SO λ IS AN EIGENVALUE OF L.

$\dim E_\lambda = 0 \Rightarrow E_\lambda = \{\vec{0}\}$, SO $\nexists$ ANY NONZERO VECTOR $\vec{v} \in E_\lambda$, I.E., $\nexists$ NONZERO $\vec{v} \in V$ WITH $L\vec{v} = \lambda\vec{v}$, SO λ IS NOT AN EIGENVALUE OF L.

8. SUPPOSE THAT $L: V \to V$ IS AN ENDOMORPHISM AND 4 IS THE ONLY EIGENVALUE OF L^2.

IF λ IS AN EIGENVALUE OF L,
THEN $\exists$ NONZERO $\vec{v} \in V$ WITH $L\vec{v} = \lambda\vec{v}$.
BUT THEN $L^2\vec{v} = L(L\vec{v}) = L(\lambda\vec{v}) = \lambda L\vec{v} = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$,
THUS λ^2 IS AN EIGENVALUE FOR L^2, SO BY HYPOTHESIS, $\lambda^2 = 4$.
$\therefore$ λ IS EITHER $+2$ OR -2.

9. IF A NONZERO VECTOR $\vec{v} \in V$ IS AN EIGENVECTOR OF V WITH EIGENVALUE 0, THEN BY DEFINITION, $L\vec{v} = 0\vec{v} = \vec{0}$, SO $\vec{v} \in \ker L$.
IF L IS INJECTIVE, THEN $\ker L = \{\vec{0}\}$, SO NO NONZERO VECTOR $\vec{v} \in V$ IS AN EIGENVECTOR WITH EIGENVALUE 0. THUS, 0 CANNOT BE AN EIGENVALUE OF AN INJECTIVE ENDOMORPHISM, I.E., AN INJECTIVE ENDOMORPHISM MUST HAVE NONZERO EIGENVALUES.

10. TAKING $\mathcal{V} = (\sin x, \sin 2x, \sin 3x)$ AND $V = \operatorname{SPAN} \mathcal{V}$, WE SAW IN PROBLEM SET 13, #4 THAT THE LINEAR TRANSFORMATION $T: V \to V$, $f(x) \mapsto 4f(x) + f''(x)$
HAS MATRIX $[T]_{\mathcal{V}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$. THIS IS DIAGONAL!

THUS $(\sin x, \sin 2x, \sin 3x)$ IS AN EIGENBASIS FOR V AND THE CORRESPONDING EIGENVALUES ARE 3, 0, AND -5, RESPECTIVELY.

DIAGONALIZATION OF MATRICES: (#11 + 12)

- THE EIGENVALUES OF AN UPPER-TRIANGULAR MATRIX A ARE JUST ITS DIAGONAL ENTRIES. (#5)
- TO FIND THE EIGENVECTORS FOR A GIVEN EIGENVALUE OF A, JUST SOLVE THE SYSTEM $[A - \lambda I]$. (#4)
- IF WE CAN FIND n LINEARLY INDEPENDENT EIGENVECTORS $(\vec{x}_1, \ldots, \vec{x}_n)$ WITH EIGENVALUES $\lambda_1, \ldots, \lambda_n$ [E.G., IF A HAS n DISTINCT EIGENVALUES (#3)], THEN $\mathcal{B} = (\vec{x}_1, \ldots, \vec{x}_n)$ MUST BE AN EIGENBASIS FOR A;
  SETTING $X = \left[\vec{x}_1 \,\vdots\, \vec{x}_2 \,\vdots\, \cdots \,\vdots\, \vec{x}_n\right]$ AND $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
  DIAGONALIZES A:
  $$[A]_{\mathcal{B}} = \Lambda, \text{ SO } A = [\mathcal{B}]\Lambda[\mathcal{B}]^{-1} = X \Lambda X^{-1}.$$

11. $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(a) BECAUSE A IS UPPER-TRIANGULAR, ITS EIGENVALUES ARE SIMPLY ITS DIAGONAL ENTRIES: $\underbrace{2, 1, 0, -1}$.

WE HAVE FOUR DISTINCT EIGENVALUES, SO WE CAN FIND ONE EIGENVECTOR FOR EACH TO OBTAIN AN EIGENBASIS:

$$\underline{\text{FOR } \lambda=2}: [A-\lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 & q_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 3 & 4 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} \text{FREE} & & & \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow q_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\text{FOR } \lambda=1}: [A-\lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 & q_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & \text{FREE} & & \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow q_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\text{FOR } \lambda=0}: [A-\lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 & q_4 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & & \text{FREE} & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow q_3 \begin{bmatrix} -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{\text{FOR } \lambda=-1}: [A-\lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} q_1 & q_2 & q_3 & q_4 \end{matrix} \\ \begin{bmatrix} 3 & 1 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & & & \text{FREE} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow q_4 \begin{bmatrix} -2/3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \left(\underset{2}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{1}{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{0}{\begin{bmatrix} -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix}}, \underset{-1}{\begin{bmatrix} -2/3 \\ 1 \\ -1 \\ 1 \end{bmatrix}} \right) \text{ IS AN EIGENBASIS FOR } \mathbb{R}^4.$$

(b) SETTING $X = \begin{bmatrix} 1 & -1 & -1/2 & -2/3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ AND $\Lambda = \begin{bmatrix} 2 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & -1 \end{bmatrix}$

WE CAN COMPUTE X^{-1}:

$$[X \mid I] = \left[\begin{array}{cccc|cccc} 1 & -1 & -1/2 & -2/3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 5/2 & 13/6 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I \mid X^{-1}]$$

AND CHECK THAT $X \Lambda X^{-1} = A$: ... ✓

12. $B = \begin{bmatrix} 3 & 1 & 5 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(a) BECAUSE A IS UPPER-TRIANGULAR, ITS EIGENVALUES ARE AGAIN JUST ITS DIAGONAL ENTRIES: $\underline{3, 1, 2}$.

WE ONLY HAVE THREE DISTINCT EIGENVALUES, SO THERE'S NO <u>GUARANTEE</u> OF AN EIGENBASIS — BUT WE CAN STILL TRY!

<u>FOR $\lambda = 3$</u>: $[B - \lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 5 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} \text{FREE} & & & \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

<u>FOR $\lambda = 1$</u>: $[B - \lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 2 & 1 & 5 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & \text{FREE} & \text{FREE} & \end{matrix} \\ \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow a_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(<u>TWO</u> BASIS VECTORS!)

<u>FOR $\lambda = 2$</u>: $[B - \lambda I] \rightsquigarrow \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 5 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & & & \text{FREE} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 25 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightsquigarrow a_4 \begin{bmatrix} -25 \\ 2 \\ 4 \\ 1 \end{bmatrix}$

$\therefore \left(\underset{3}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{1}{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{1}{\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{2}{\begin{bmatrix} -25 \\ 2 \\ 4 \\ 1 \end{bmatrix}} \right)$ IS AN EIGENBASIS FOR $\mathbb{R}^4$.

(b) SETTING $X = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & -25 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ AND $\Lambda = \begin{bmatrix} 3 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 2 \end{bmatrix}$,

WE CAN COMPUTE X^{-1}:

$$[X \mid I] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & -\frac{1}{2} & -\frac{5}{2} & -25 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{5}{2} & 14 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I \mid X^{-1}]$$

NOTE THAT THIS VERIFIES THAT WE DO, INDEED, HAVE A BASIS!

AND CHECK THAT $X \Lambda X^{-1} = B$: ... ✓