1. [V: VECTOR SPACE]

(a) AN ENDOMORPHISM OF V IS A LINEAR TRANSFORMATION $L: V \to V$, I.E., A L.T. FROM V INTO ITSELF.
"ENDO"

(b) AN ENDOMORPHISM $L: V \to V$ CAN BE COMPOSED WITH ITSELF, E.G., $L^2 = L \circ L : V \to V$, $L^3 = L \circ L \circ L$, ETC.

(c) BECAUSE THE DOMAIN AND CODOMAIN OF AN ENDOMORPHISM ARE THE SAME VECTOR SPACE, WHEN WE REPRESENT AN ENDOMORPHISM BY A MATRIX (AND WHEN WE CHANGE BASES), WE TAKE THE SAME BASIS FOR BOTH ITS DOMAIN AND ITS CODOMAIN AND DON'T REPEAT IT NOTATIONALLY.
THAT IS:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, shift left, "{[v]^{-1}}"] & V \arrow[d, shift left, "{[v]^{-1}}"] \\ \mathbb{R}^n \arrow[r, "{[L]_v}"'] \arrow[u, shift left, "{[v]}"] & \mathbb{R}^n \arrow[u, shift left, "{[v]}"] \end{tikzcd}$$

$$\underline{[L]_v = [v]^{-1} L [v]}$$

I.E.,

$$\underline{L = [v][L]_v[v]^{-1}}$$

CHANGE OF BASIS BECOMES:
$$\underset{n\times n}{[L]_{v'}} = \Big(\underbrace{\underset{n\times n}{[v']^{-1}[v]}}\Big)\underset{n\times n}{[L]_v}\Big(\underbrace{\underset{n\times n}{[v]^{-1}[v']}}\Big)$$

THE CHANGE-OF-BASIS MATRICES ARE INVERSES OF ONE ANOTHER!

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "{[L]_{v'}}"] \arrow[dd, bend right=60, "{[v]^{-1}[v']}"'] \arrow[d, shift left, "{[v']}"] & \mathbb{R}^n \arrow[d, shift left, "{[v']}"] \\ V \arrow[r, "L"] \arrow[u, shift left, "{[v']^{-1}}"] \arrow[d, shift left, "{[v]^{-1}}"] & V \arrow[u, shift left, "{[v']^{-1}}"] \arrow[d, shift left, "{[v]^{-1}}"] \\ \mathbb{R}^n \arrow[r, "{[L]_v}"'] \arrow[u, shift left, "{[v]}"] & \mathbb{R}^n \arrow[u, shift left, "{[v]}"] \arrow[uu, bend right=60, "{[v']^{-1}[v]}"'] \end{tikzcd}$$

(d) TWO $n\times n$ MATRICES A, B ARE SIMILAR IF THERE IS AN INVERTIBLE $n \times n$ MATRIX X SUCH THAT $B = XAX^{-1}$;
WE DENOTE THE STATEMENT THAT A IS SIMILAR TO B BY "$A \sim B$".

2. [$L: V \to V$, v: BASIS FOR V]

(a) $[L^k]_v = ([L]_v)^k$; (THE MATRIX FOR THE k^{TH} ITERATE OF L IS THE k^{TH} ITERATE OF THE MATRIX FOR L)

BECAUSE
$$([L]_v)^k = \underbrace{[L]_v[L]_v \cdots [L]_v}_{k \text{ TIMES}}$$
$$= ([v]^{-1}L[v])([v]^{-1}L[v]) \cdots ([v]^{-1}L[v])$$
$$= [v]^{-1}L\cancel{[v][v]^{-1}}L\cancel{[v][v^{-1}]} \cdots \cancel{[v][v^{-1}]}L[v]$$
$$= [v]^{-1} L \circ L \circ \cdots \circ L[v]$$
$$= [v]^{-1}L^k[v] = [L^k]_v$$

(b) IF L IS AN AUTOMORPHISM (SO L^{-1} EXISTS),
THEN $[L^{-1}]_v = ([L]_v)^{-1}$;

BECAUSE
$$[L^{-1}]_v[L]_v = ([v]^{-1}L^{-1}[v])([v]^{-1}L[v])$$
$$= [v]^{-1}L^{-1}\cancel{[v][v]^{-1}}L[v]$$
$$= [v]^{-1}\cancel{L^{-1}L}[v] = \cancel{[v]^{-1}[v]} = I$$

AND SIMILARLY FOR $[L]_v[L^{-1}]_v$.

(TO BE THE INVERSE OF $[L]_v$ MEANS THAT YOUR COMPOSITION WITH IT ON EACH SIDE CANCELS)

3. AN ENDOMORPHISM $L: V \to V$ IS DIAGONALIZABLE IF THERE IS A BASIS $\mathcal{V}$ OF V FOR WHICH $[L]_{\mathcal{V}}$ IS A DIAGONAL MATRIX.

(I.E., A MATRIX OF THE FORM $\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$)

A DIAGONALIZATION OF L GIVES US A VERY SIMPLE DESCRIPTION OF L: IF $\mathcal{V} = (\vec{v}_1, \dots, \vec{v}_n)$ AND $[L]_{\mathcal{V}} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$,

THEN $L\vec{v}_i \leftrightarrow \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\vec{e}_i$

$= i^{TH}$ COLUMN OF $\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$

$= \begin{bmatrix}0\\ \vdots \\ \lambda_i \\ \vdots \\ 0\end{bmatrix} = \lambda_i \vec{e}_i \longleftrightarrow \lambda_i \vec{v}_i$

(I.E., L SIMPLY SCALES EACH BASIS VECTOR $\vec{v}_i$ BY A FACTOR OF λ_i.)

4. TWO ENDOMORPHISMS $A, B: V \to V$ COMMUTE IF $AB = BA$.

IN $\mathbb{R}^2$, $\underset{A}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\underset{B}{\begin{bmatrix}1&0\\0&0\end{bmatrix}} = \begin{bmatrix}0&0\\1&0\end{bmatrix}$,

$\neq$

WHILE $\underset{B}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}\underset{A}{\begin{bmatrix}0&1\\1&0\end{bmatrix}} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$, SO THESE MATRICES DO NOT COMMUTE.

* ENDOMORPHISMS DO NOT, IN GENERAL, COMMUTE!

5. [A, B, C : $n \times n$ MATRICES]

(a) CLAIM: $A \sim A$. $\to$ I.E., $\exists$ INVERTIBLE MATRIX X WITH $A = XAX^{-1}$

PROOF: TAKE $X = I_n$. THEN X IS INVERTIBLE AND

$XAX^{-1} = \cancel{I_n} A I_n^{-1} = A \cancel{I_n} = A$ ■

(b) CLAIM: IF $A \sim B$, THEN $B \sim A$.

PROOF: SUPPOSE THAT $A \sim B$,

I.E., $\exists$ INVERTIBLE MATRIX X WITH $B = XAX^{-1}$. *

[GOAL: $B \sim A$, I.E., $\exists$ INVERTIBLE MATRIX Y WITH $A = YBY^{-1}$]

$B = XAX^{-1}$

$X^{-1}BX = X^{-1}XAX^{-1}X$

$= A$

... TAKE $Y = X^{-1}$!

TAKE $Y = X^{-1}$; THEN Y IS INVERTIBLE,

AND $YBY^{-1} = Y(XAX^{-1})Y^{-1}$ FROM *

$= \cancel{X^{-1}X} A \cancel{X^{-1}X} = A$ ■

(c) CLAIM: IF $A \sim B$ AND $B \sim C$, THEN $A \sim C$.

PROOF: SUPPOSE THAT $A \sim B$ AND $B \sim C$,

I.E., $\exists$ INVERTIBLE MATRIX X WITH $B = XAX^{-1}$ (*)

AND $\exists$ INVERTIBLE MATRIX Y WITH $C = YBY^{-1}$ (**)

[GOAL: $A \sim C$, I.E., $\exists$ INVERTIBLE MATRIX Z WITH $C = ZAZ^{-1}$]

$C = YBY^{-1}$

$= (YX)A(X^{-1}Y^{-1})$

... TAKE $Z = YX$!

TAKE $Z = YX$; THEN Z IS INVERTIBLE, AND

$ZAZ^{-1} = (YX)A(YX)^{-1} = YXAX^{-1}Y^{-1}$

$= Y(XAX^{-1})Y^{-1}$

$= YBY^{-1}$ FROM *

$= C$ FROM ** ■

(d) CLAIM: $A \sim I_n \iff A = I_n$

PROOF: $A \sim I_n \Rightarrow A = I_n$: [GOAL: $A = I_n$]

SUPPOSE THAT $A \sim I_n$,

I.E., $\exists$ INVERTIBLE MATRIX X WITH $I_n = XAX^{-1}$.

THEN $X^{-1}\cancel{I_n}X = X^{-1}(XAX^{-1})X$

SO $\cancel{X^{-1}}X = (\cancel{X^{-1}}X)\,A\,(\cancel{X^{-1}}X)$, I.E., $I_n = A$ ✓

$A = I_n \Rightarrow A \sim I_n$: (COULD JUST APPLY PART (a))

SUPPOSE THAT $A = I_n$.

[GOAL: $A \sim I_n$,
I.E., $\exists$ INVERTIBLE MATRIX X WITH $I_n = XAX^{-1}$]

TAKE $X = I_n$; THEN X IS INVERTIBLE,

AND $XAX^{-1} = I_n I_n \cancel{I_n^{-1}} = I_n$ ✓ ■

$XI_nX^{-1} = XX^{-1} = I_n$ FOR ANY INVERTIBLE X; E.G., TAKE $X = I_n$.

6. $L: V \to V$ ENDOMORPHISM; $\mathcal{V}$: BASIS FOR V WITH $[L]_{\mathcal{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

(a) $L\vec{v}_1 = [\mathcal{V}][L]_{\mathcal{V}}[\mathcal{V}]^{-1}\vec{v}_1 = [\mathcal{V}]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\vec{e}_1 = [\mathcal{V}]\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1\vec{v}_1$.

$L\vec{v}_2 = [\mathcal{V}][L]_{\mathcal{V}}[\mathcal{V}]^{-1}\vec{v}_2 = [\mathcal{V}]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\vec{e}_2 = [\mathcal{V}]\begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_2\vec{v}_2$

$\vdots$

$L\vec{v}_n = [\mathcal{V}][L]_{\mathcal{V}}[\mathcal{V}]^{-1}\vec{v}_n = [\mathcal{V}]\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\vec{e}_n = [\mathcal{V}]\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{bmatrix} = \lambda_n\vec{v}_n$

(b) FROM 2(a), $[L^2]_{\mathcal{V}} = ([L]_{\mathcal{V}})^2$

$= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix}$

SIMILARLY, $[L^k]_{\mathcal{V}} = ([L]_{\mathcal{V}})^k$

$= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$.

(c) L IS INVERTIBLE $\iff [L]_{\mathcal{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ IS INVERTIBLE

$\iff$ ALL λ_i's ARE NONZERO. (NEED n PIVOTS)

IN THIS CASE, $[L^{-1}]_{\mathcal{V}} = \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{bmatrix}$, BECAUSE FROM 2(b),

$[L^{-1}]_{\mathcal{V}} = [L]_{\mathcal{V}}^{-1}$, AND $\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I_n$.

(ALTERNATIVELY, COMPUTE $[L]_{\mathcal{V}}^{-1}$ BY REDUCTION: $\left[\begin{array}{ccc|ccc} \lambda_1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & \lambda_n & & & 1 \end{array}\right]$
$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & & & \lambda_1^{-1} & & \\ & \ddots & & & \ddots & \\ & & 1 & & & \lambda_n^{-1} \end{array}\right]$)

7. IF A, B ARE ENDOMORPHISMS OF V, THEN:

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B)$$
$$= AA + AB + BA + BB = \underline{A^2 + AB + BA + B^2}.$$

(IF $A + B$ COMMUTE, THEN THIS IS $A^2+AB+AB+B^2 = A^2+2AB+B^2$)

$$(A+B)^3 = (A+B)(A+B)^2 = (A+B)(A^2+AB+BA+B^2)$$
$$= A(A^2+AB+BA+B^2) + B(A^2+AB+BA+B^2)$$
$$= \underline{A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3}$$

(IF $A + B$ COMMUTE, THEN THIS IS $A^3 + 3A^2B + 3AB^2 + B^3$).

8. [A IS NILPOTENT IF $\exists\, k>0$ SUCH THAT $A^k = 0$]

ZERO POWER

(a) CLAIM: IF A IS NILPOTENT AND $A \sim B$, THEN B IS NILPOTENT.

PROOF: SUPPOSE THAT A IS NILPOTENT AND $A \sim B$,

I.E., $\exists\, k>0$ SUCH THAT $A^k = 0$ (*)

AND $\exists$ INVERTIBLE MATRIX X WITH $B = XAX^{-1}$. (**)

[GOAL: $\exists\, k>0$ SUCH THAT $B^k = 0$]

TAKE $k>0$ AS IN THE HYPOTHESIS *, AND X AS IN **.

THEN $B^k = (XAX^{-1})^k$ FROM **

$$= \underbrace{(XAX^{-1})(XAX^{-1})\dots(XAX^{-1})}_{k \text{ TIMES}}$$
$$= XA\cancel{X^{-1}}\cancel{X}A\cancel{X^{-1}}\cancel{X}\dots\cancel{X^{-1}}\cancel{X}AX^{-1}$$
$$= XA^kX^{-1}$$
$$= X0X^{-1} \quad \text{FROM } *$$
$$= X0 = 0,$$

SO BY DEFINITION, B IS NILPOTENT. ■

(b) SUPPOSE THAT A IS NILPOTENT + DIAGONALIZABLE.

THEN $\exists\, k>0$ SUCH THAT $A^k = 0$, AND

$\exists$ BASIS $\mathcal{B}$ WITH $[A]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

BUT THEN $[A^k]_{\mathcal{B}} = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$ AND $[A^k]_{\mathcal{B}} = [0]_{\mathcal{B}} = 0$,

SO WE MUST HAVE $\lambda_1^k, \dots, \lambda_n^k = 0$, I.E., $\lambda_1, \dots, \lambda_n = 0$.

BUT THEN $[A]_{\mathcal{B}} = 0$, SO A MUST BE THE ZERO L.T.

(c) CLAIM: IF A IS NILPOTENT, THEN $I - A$ IS INVERTIBLE.

PROOF: SUPPOSE THAT A IS NILPOTENT,

I.E., $\exists\, k>0$ SUCH THAT $A^k = 0$.

{GOAL: TO FIND AN INVERSE FOR A,
I.E., A MATRIX B WITH $B(I-A) = I = (I-A)B$

HINT: $\frac{1}{1-x} = 1 + x + x^2 + \dots$ MEANS THAT THE "INVERSE" OF $1-x$ IS $1+x+x^2+\dots$

LET'S TRY THIS WITH A:

INVERSE OF $I-A \stackrel{?}{=} I + A + A^2 + \dots$
$= I + A + A^2 + \dots + A^{k-1} + \underline{A^k + \dots}$
$= I + A + A^2 + \dots + A^{k-1}$! ALL 0

LET $B = I + A + A^2 + \dots + A^{k-1}$.

THEN $B(I-A) = (I + A + A^2 + \dots + A^{k-1})(I-A)$
$$= (I + A + A^2 + \dots + A^{k-1})I - (I + A + A^2 + \dots + A^{k-1})A$$
$$= I + \cancel{A} + \cancel{A^2} + \dots + \cancel{A^{k-1}} - \cancel{A} - \cancel{A^2} - A^3 - \dots - A^k$$

AND, SIMILARLY,

$(I-A)B = (I-A)(I + A + A^2 + \dots + A^{k-1})$
$$= I(I + A + A^2 + \dots + A^{k-1}) - A(I + A + A^2 + \dots + A^{k-1})$$
$$= I + \cancel{A} + \cancel{A^2} + \dots + \cancel{A^{k-1}} - \cancel{A} - \cancel{A^2} - A^3 - \dots - A^k$$
$$= I - A^k = I - 0 = I \checkmark$$

THUS, BY DEFINITION, B IS THE INVERSE OF $I - A$,

SO $I - A$ IS INVERTIBLE ■

9. [A HAS FINITE ORDER MEANS $\exists\, k>0$ SUCH THAT $A^k = I$.]

(a) CLAIM: IF A HAS FINITE ORDER AND $A \sim B$, THEN B HAS FINITE ORDER.

PROOF: SUPPOSE THAT A HAS FINITE ORDER AND $A \sim B$, I.E., $\exists\, k>0$ SUCH THAT $A^k = I$ $(*)$

AND $\exists$ INVERTIBLE MATRIX X WITH $B = XAX^{-1}$. $(**)$

[GOAL: B HAS FINITE ORDER, I.E., $\exists\, k>0$ SUCH THAT $B^k = I$]

TAKE $k>0$ AS IN $(*)$ AND X AS IN $(**)$.

THEN $B^k = (XAX^{-1})^k$ FROM $**$

$$= (XAX^{-1})(XAX^{-1})\cdots(XAX^{-1})$$

$$= XA\cancel{X^{-1}X}A\cancel{X^{-1}X}\cdots\cancel{X^{-1}X}AX^{-1}$$

$$= XA^kX^{-1} = XIX^{-1} \quad \text{FROM } *$$

$$= XX^{-1} = I,$$

SO BY DEFINITION, B HAS FINITE ORDER. ■

(b) CLAIM: IF A HAS FINITE ORDER, THEN A IS INVERTIBLE.

PROOF: SUPPOSE THAT A HAS FINITE ORDER, I.E., $\exists\, k>0$ SUCH THAT $A^k = I$

[GOAL: TO FIND AN INVERSE FOR A, I.E., A MATRIX B WITH $BA = I = AB$]

WELL, $A^k = I$ $\Rightarrow A(A^{k-1}) = I$ AND $(A^{k-1})A = I$ ... TAKE $B = A^{k-1}$

LET $B = A^{k-1}$! $(A^0 = I)$.

THEN $BA = (A^{k-1})A = A^k = I$ BY HYPOTHESIS,

AND $AB = A(A^{k-1}) = A^k = I$ BY HYPOTHESIS.

BY DEFINITION, B IS THE INVERSE OF A, SO A IS INVERTIBLE. ■

(c) SUPPOSE THAT A HAS FINITE ORDER AND IS DIAGONALIZABLE, I.E., $\exists\, k>0$ SUCH THAT $A^k = I$, AND A BASIS $\mathcal{B}$ WITH $[A]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

THEN $[A^k]_{\mathcal{B}} = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}$ AND $[A^k]_{\mathcal{B}} = [I]_{\mathcal{B}} = I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$,

SO $\lambda_1^k, \ldots, \lambda_n^k = 1$. (IN THE CASE OF REAL NUMBERS, EACH λ_i MUST BE 1 (OR -1, IF k IS EVEN))