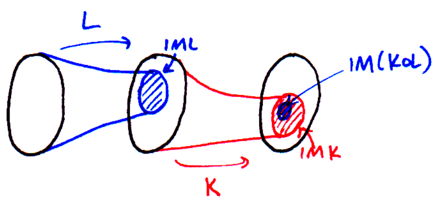
1. IF A LINEAR TRANSFORMATION $L: V \to W$ IS INVERTIBLE, THEN L MUST BE BIJECTIVE, SO NULLITY $L = 0$ AND RANK $L = \dim W$. BUT IF L IS A BIJECTIVE L.T., THEN IT IS AN ISOMORPHISM, SO $V \cong W$ AND THUS $\dim V = \dim W$.
IN SUMMARY, RANK $L = \dim V = \dim W$, AND NULLITY $L = 0$.

2. $L: V \to W$ AND $K: W \to U$: LINEAR TRANSFORMATIONS



(a) CLAIM: $IM(K \circ L) \subset IM\, K$ $\longrightarrow$ I.E., $\vec{u} \in IM(K \circ L) \Rightarrow \vec{u} \in IM\, K$

PROOF: SUPPOSE THAT $\vec{u} \in IM(K \circ L)$

[RECALL THAT $IM(K \circ L) = \{ (K \circ L)(\vec{v}) : \vec{v} \in V \}$]

THEN $\vec{u} = (K \circ L)(\vec{v})$, WHERE $\vec{v} \in V$

SO BY DEFINITION, $\vec{u} = K(L(\vec{v}))$ $*$

[GOAL: $\vec{u} \in IM\, K = \{ K\vec{w} : \vec{w} \in W \}$ — NEED TO FIND $\vec{w}$!

LET $\vec{w} = L(\vec{v}) \in W$.

THEN $K\vec{w} = K(L(\vec{v})) = \vec{u}$ FROM $*$,

SO BY DEFINITION, $\vec{u} \in IM\, K$. ∎

COROLLARY: RANK $(K \circ L) \le$ RANK K

PROOF: SINCE $IM(K \circ L)$ IS A SUBSPACE OF $IM\, K$,
$\dim(IM(K \circ L)) \le \dim(IM\, K)$,
SO BY DEFINITION OF RANK, RANK $(K \circ L) \le$ RANK K. ∎

(b) CLAIM: RANK $(K \circ L) \le$ RANK L

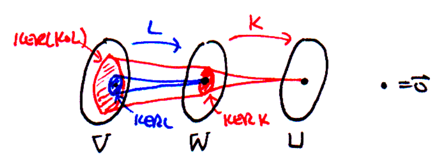
PROOF: LET $\{\vec{k}_1, \dots, \vec{k}_n ; \vec{v}_1, \dots, \vec{v}_r\}$ BE A BASIS FOR V,
WHERE $\{\vec{k}_1, \dots, \vec{k}_n\}$ ARE A BASIS FOR KER L AND $r =$ RANK L,
AS IN THE PROOF OF THE RANK + NULLITY THEOREM.
THEN SINCE THIS COLLECTION SPANS V, THE COLLECTION

$$\{(K \circ L)(\vec{k}_1), \dots, (K \circ L)(\vec{k}_n) ; (K \circ L)(\vec{v}_1), \dots, (K \circ L)(\vec{v}_r)\}$$

SPANS $IM(K \circ L)$.

BUT $\forall i$, $(K \circ L)(\vec{k}_i) = K(L(\vec{k}_i))$
$= K(\vec{0})$ SINCE $\vec{k}_i \in$ KER L
$= \vec{0}$

SO $\{(K \circ L)(\vec{v}_1), \dots, (K \circ L)(\vec{v}_r)\}$ SPANS $IM(K \circ L)$,
AND THUS CAN BE REDUCED TO A BASIS FOR $IM(K \circ L)$;
THUS, $\dim(IM(K \circ L)) \le r =$ RANK L,
SO BY DEFINITION, RANK $(K \circ L) \le$ RANK L ∎



(c) CLAIM: KER $L \subset$ KER $(K \circ L)$ $\longrightarrow$ I.E., $\vec{v} \in$ KER $L \Rightarrow \vec{v} \in$ KER $(K \circ L)$

PROOF: LET $\vec{v} \in$ KER L; THEN BY DEFINITION, $L(\vec{v}) = \vec{0}_W$.

[GOAL: $\vec{v} \in$ KER$(K \circ L)$, I.E., $(K \circ L)(\vec{v}) = \vec{0}_U$]

THUS, $(K \circ L)(\vec{v}) \overset{DEF}{=} K(L(\vec{v})) = K(\vec{0}_W) = \vec{0}_U$. ∎

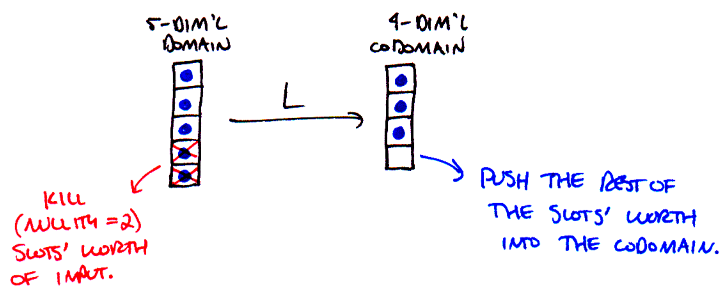
COROLLARY: NULLITY $(K \circ L) \ge$ NULLITY L

PROOF: SINCE KER L IS A SUBSPACE OF KER$(K \circ L)$,
$\dim(\text{KER } L) \le \dim(\text{KER}(K \circ L))$, SO BY DEFINITION,
NULLITY $L \le$ NULLITY $(K \circ L)$. ∎

3. WE THINK OF AN n-DIMENSIONAL VECTOR SPACE AS HAVING "SIZE" n (FORMALLY, THIS IS JUSTIFIED BY COUNTING THE NUMBER OF VECTORS IN ANY BASIS FOR IT).

WE'VE SEEN IN OUR PROOF OF THE RANK + NULLITY THEOREM THAT ANY LINEAR TRANSFORMATION FROM ONE SUCH SPACE TO ANOTHER KILLS ITS NULLITY'S WORTH OF THESE AND GENERATES AN IMAGE OF SIZE GIVEN BY ITS RANK.

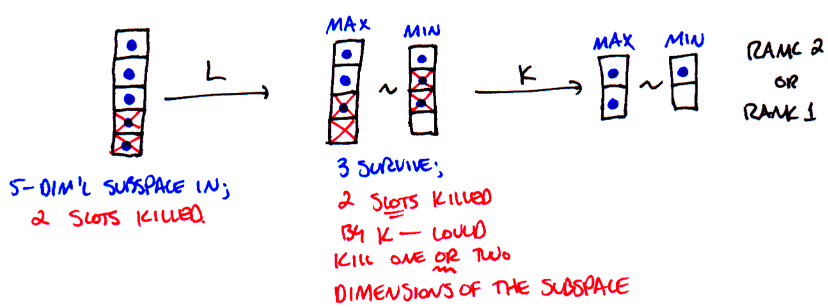
E.G., FOR $L: \mathbb{R}^5 \to \mathbb{R}^4$ OF RANK 3 ($\therefore$ NULLITY 2),



FOLLOWING THE "SIZE" OF THE SUBSPACES AS THEY'RE PUSHED THROUGH GIVES US THE RANK OF A COMPOSITION; COUNTING HOW MANY ARE KILLED GIVES US THE NULLITY.

✱ KEY: A L.T. WILL KILL ITS NULLITY'S WORTH OF SLOTS!

E.G., $K \circ L$, WHERE L IS AS ABOVE AND $K: \mathbb{R}^4 \to \mathbb{R}^2$ HAS RANK 2:



NET EFFECT: $K \circ L$ MUST HAVE RANK 2 OR 1.

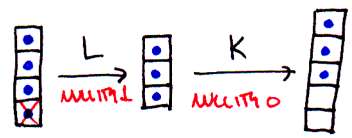
(KEEPING TRACK OF THE EXTREME CASES SHOWS YOU THE POSSIBLE RANGE OF DIMENSIONS)

4. (a) (JUST USE RANK + NULLITY = DIM OF DOMAIN)

- NULLITY $L = 4 - 3 = 1$
- NULLITY $K = 3 - 3 = 0$
- NULLITY $M = 5 - 1 = 4$
- NULLITY $N = 4 - 4 = 0$
- NULLITY $R = 3 - 3 = 0$
- NULLITY $J = 5 - 4 = 1$

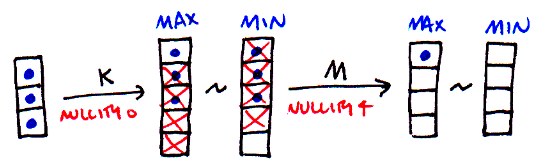
(b)
- L IS NOT INJECTIVE (NULLITY $\neq 0$), BUT IS SURJECTIVE (RANK $= 3$)
- K IS INJECTIVE (NULLITY $= 0$), BUT NOT SURJECTIVE (RANK $= 3 < 5$)
- M IS NOT INJECTIVE (NULLITY $\neq 0$) AND NOT SURJECTIVE (RANK $= 1 < 4$)
- N IS BIJECTIVE (NULLITY $= 0$, RANK $= 4$)
- R IS BIJECTIVE (NULLITY $= 0$, RANK $= 3$)
- J IS NOT INJECTIVE (NULLITY $\neq 0$) AND NOT SURJECTIVE (RANK $= 4 < 5$)

(c) (i)



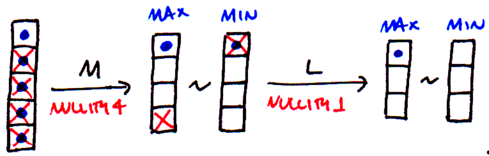
$K \circ L$ MUST HAVE RANK 3 ($\therefore$ NULLITY 1)
$\therefore$ CAN BE NEITHER INJECTIVE NOR SURJECTIVE.

(ii)



$M \circ K$ EITHER HAS RANK 1 ($\therefore$ NULLITY 2) OR RANK 0 ($\therefore$ NULLITY 3)
$\therefore$ CAN BE NEITHER INJECTIVE NOR SURJECTIVE.

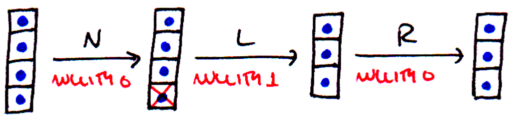
(iii)



L∘M EITHER HAS RANK 1 (∴ NULLITY 4) OR RANK 0 (∴ NULLITY 5)

∴ CAN BE NEITHER INJECTIVE NOR SURJECTIVE

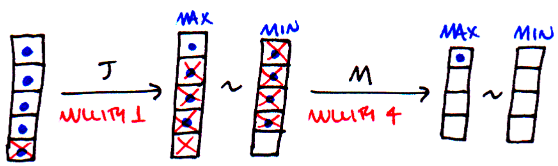
(iv)



R∘L∘N HAS RANK 3 (∴ NULLITY 1)

∴ MUST BE SURJECTIVE, BUT NOT INJECTIVE.

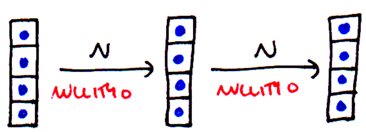
(v)



M∘J EITHER HAS RANK 1 (∴ NULLITY 4) OR RANK 0 (∴ NULLITY 5)

∴ CAN BE NEITHER INJECTIVE NOR SURJECTIVE

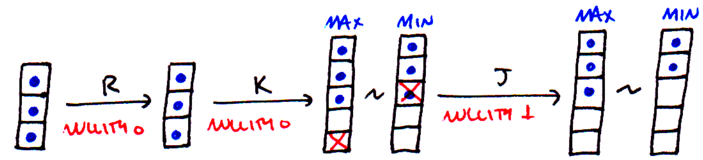
(vi)



N∘N MUST HAVE RANK 4 (∴ NULLITY 0)

∴ MUST BE BIJECTIVE.

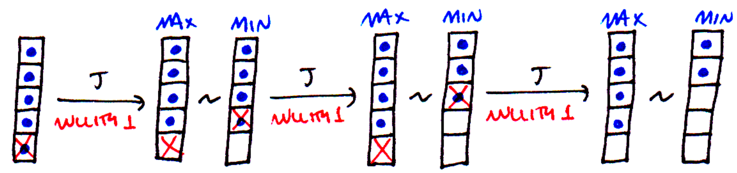
(vii)



J∘K∘R EITHER HAS RANK 3 (∴ NULLITY 0, SO INJECTIVE) OR RANK 2 (∴ NULLITY 1, SO NOT INJECTIVE)

∴ CAN'T BE SURJECTIVE.

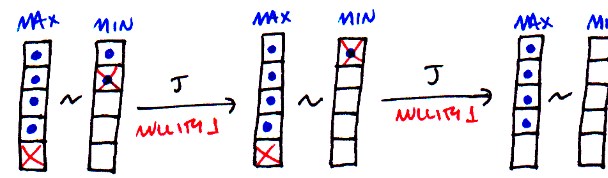
(viii)



J∘J∘J EITHER HAS RANK 4 (∴ NULLITY 1), RANK 3 (∴ NULLITY 2), OR RANK 2 (∴ NULLITY 3)

∴ CAN'T BE INJECTIVE, CAN'T BE SURJECTIVE.

(ix) (PICKING UP WHERE WE LEFT OFF AT J∘J∘J)



J∘J∘J∘J∘J EITHER HAS RANK 4 (∴ NULLITY 1), RANK 3 (∴ NULLITY 2), RANK 2 (∴ NULLITY 3), RANK 1 (∴ NULLITY 4), OR RANK 0 (∴ NULLITY 5).

∴ CAN'T BE INJECTIVE, CAN'T BE SURJECTIVE.

5. A: $m \times n$ MATRIX; B: $n \times p$ MATRIX

(a) AS A L.T., $\underset{\text{DOMAIN}}{A: \mathbb{R}^n} \to \underset{\text{CODOMAIN}}{\mathbb{R}^m}$ AND $\underset{\text{DOMAIN}}{B: \mathbb{R}^p} \to \underset{\text{CODOMAIN}}{\mathbb{R}^n}$.

(b) THE MATRIX AB SIMPLY MEANS THE MATRIX FOR THE CORRESPONDING COMPOSITION $A \circ B: \mathbb{R}^p \to \mathbb{R}^m$.

THUS, AB IS AN $m \times p$ MATRIX, AND OUR RULE FOR COMPUTING IT SIMPLY COMES FROM ANALYZING THE COMPOSITION $A \circ B$ AS A LINEAR TRANSFORMATION.

(c) THE j^{TH} COLUMN OF AB

$= AB\vec{e}_j$

$= A(B\vec{e}_j)$ BECAUSE AB IS A COMPOSITION

$= A$ (j^{TH} COLUMN OF B)

$=$ L.C. OF COL'S OF A, WITH SCALARS GIVEN BY THE j^{TH} COL. OF B.

E.G., $$\begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 & 0 \\ 2 & 3 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 13 & -2 \\ 9 & 11 & 24 & -8 \\ 3 & -3 & 12 & 0 \end{bmatrix}$$

6. THE IDENTITY MATRIX OF SIZE n IS $I_n = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ \vdots & 0 & 1 & & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & & & 1 \end{bmatrix}}$,

WHICH CORRESPONDS TO THE STANDARD BASIS FOR $\mathbb{R}^n$.

THIS MATRIX GIVES US THE IDENTITY L.T. $: \mathbb{R}^n \to \mathbb{R}^n$, $I_n\vec{x} = \vec{x}$,

SO
- $I_n\vec{x} = \vec{x}$ $\forall\, x \in \mathbb{R}^n$
- $AI_n = A$ FOR ANY $m \times n$ MATRIX A
- $I_nB = B$ FOR ANY $n \times p$ MATRIX B.

* I_n EVAPORATES ANYWHERE THAT IT ACTS ON A VECTOR OR IS COMPOSED WITH ANOTHER MATRIX!

7. (a) AN INVERSE FOR A MATRIX A IS A MATRIX A^{-1} FOR WHICH $A^{-1}A = I = AA^{-1}$ (WHERE I IS THE IDENTITY MATRIX OF THE APPROPRIATE SIZE).

(I.E., IT'S THE INVERSE FUNCTION FOR THE L.T. GIVEN BY A)

A IS CALLED INVERTIBLE IF IT HAS AN INVERSE.

(b) IN ORDER FOR A TO BE INVERTIBLE, IT MUST BE SQUARE, WITH "FULL RANK" AND ZERO NULLITY.

I.E. RANK AS BIG AS POSSIBLE — HERE, THIS MEANS EQUAL TO THE # OF ROWS (OR COLUMNS) OF A.

(c) USING THE FACT THAT $AA^{-1} = I$, WE CAN FIND THE j^{TH} COLUMN OF A^{-1} AS FOLLOWS:

$$AA^{-1}\vec{e}_j = I\vec{e}_j = \vec{e}_j, \text{ SO } A(A^{-1}\vec{e}_j) = \vec{e}_j$$

$$\text{I.E., } A(j^{\text{TH}} \text{ COLUMN OF } A^{-1}) = \vec{e}_j,$$

SO THE j^{TH} COLUMN OF A^{-1} IS SIMPLY THE SOLUTION OF THE SYSTEM $[A \mid \vec{e}_j]$.

WE CAN THUS FIND ALL COLUMNS OF A^{-1} AT ONCE BY REDUCING THE MULTIPLY-AUGMENTED SYSTEM $[A \mid I]$:

$$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$$

ANALYZING THE RESULT SHOWS US THAT THE COLUMNS HERE ARE PRECISELY THOSE OF A^{-1}!

SINCE A IS SQUARE + HAS FULL RANK, THE LEFT SIDE MUST REDUCE TO THIS!

* IF WE TRY THIS FOR A NON-INVERTIBLE MATRIX A, THE LEFT SIDE WON'T COME OUT TO I, AND THE SYSTEM WILL BE INCONSISTENT IN SOME COLUMN(S).

(d) CLAIM: IF $AA^{-1} = I$ AND $BA = I$, THEN $B = A^{-1}$

PROOF: SUPPOSE THAT $AA^{-1} = I$ AND $BA = I$.
THEN RIGHT-COMPOSING $BA = I$ WITH A^{-1},
WE HAVE $BAA^{-1} = IA^{-1}$
SO $BI = A^{-1}$ BY THE HYPOTHESIS THAT $AA^{-1} = I$
$\therefore$ $B = A^{-1}$ BY PROPERTIES OF I ■

8. IF WE HAVE AN INVERSE A^{-1} FOR A, THEN

$A\vec{x} = \vec{b} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{b}$ (COMPOSING WITH A^{-1} ON THE LEFT)

$\Rightarrow (A^{-1}A)\vec{x} = A^{-1}\vec{b}$

$\Rightarrow I\vec{x} = A^{-1}\vec{b}$ BY DEFINITION OF INVERSE

$\Rightarrow \vec{x} = A^{-1}\vec{b}$ BY PROPERTIES OF I

— I.E., JUST LET A^{-1} ACT ON $\vec{b}$

* HOWEVER, THE INITIAL "IF" MAKES THIS AN EXTREMELY SPECIALIZED APPROACH: IT ONLY WORKS FOR "SQUARE" SYSTEMS, HAVING THE SAME NUMBER OF EQUATIONS + UNKNOWNS, AND IT ONLY WORKS IF THE COEFFICIENT MATRIX IS INVERTIBLE!