1. VIA THE F.T.L.T., ANY COMPUTATIONAL QUESTION INVOLVING ABSTRACT LINEAR TRANSFORMATIONS [BETWEEN FINITE-DIMENSIONAL VECTOR SPACES] CAN BE TRANSLATED INTO A QUESTION ABOUT COLUMN VECTORS AND MATRICES; TO DO SO, WE SIMPLY NEED BASES FOR THE DOMAIN AND CODOMAIN OF THE LINEAR TRANSFORMATION IN QUESTION.

   * WHEN FINISHED, OUR ANSWER(S) SHOULD BE TRANSLATED BACK TO THE VECTOR SPACES WHENCE THE PROBLEM STARTED!

2. $T: P_3(x) \longrightarrow P_3(x)$, $\quad f(x) \longmapsto xf''(x) + f'(x)$

   TAKE BASES $\mathcal{V}, \mathcal{W} = (1, x, x^2, x^3)$

(a) $1 \overset{T}{\longmapsto} 0$

$x \longmapsto 1$

$x^2 \longmapsto 2x + 2x = 4x$

$x^3 \longmapsto x \cdot 6x + 3x^2 = 9x^2$

$$\therefore\ {}_{\mathcal{W}}[T]_{\mathcal{V}} = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = A$$

A REDUCES TO

$$\begin{array}{c} \begin{array}{cccc} \text{FREE} & & & \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

a_1: FREE

$a_2 = 0$

$a_3 = 0$

$a_4 = 0$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

SO A BASIS FOR $N(A)$ IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

AND A BASIS FOR $C(A)$ IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 9 \\ 0 \end{bmatrix} \right\}$

$\therefore \{1\}$ IS A BASIS FOR KER T,

AND $\{1, 4x, 9x^2\}$ IS A BASIS FOR IM T.

- RANK $L = 3$, NULLITY $L = 1$: T IS NEITHER INJECTIVE NOR SURJECTIVE
  (NULLITY $\neq 0$) (RANK $\neq 4$)

(b) (JUST TRANSLATE INTO A LINEAR SYSTEM!)

$$\begin{array}{r} \\ \cdot\frac{1}{4} \rightarrow \\ \cdot\frac{1}{9} \rightarrow \\ \\ \end{array} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{c} \begin{array}{ccccc} \text{FREE} & & & & \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 9 & 1/9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

a_1: FREE

$a_2 = 1$

$a_3 = 1/4$

$a_4 = 1/9$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ 1 \\ 1/4 \\ 1/9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1/4 \\ 1/9 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$T(f(x)) = 1 + x + x^2$

$\Rightarrow f(x) = \underbrace{\left(x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3\right)}_{\text{THIS}} + \underbrace{a_1}_{\text{ANYTHING IN KER } T}$

THIS PLUS ANYTHING IN KER T

BASES α, ω

3. $V = \mathrm{SPAN}(e^x, xe^x, x^2e^x, x^3e^x) \subset C(\mathbb{R})$

$L: V \to V, \quad f(x) \mapsto f''(x) - 2f'(x) + f(x).$

(a) $e^x \overset{L}{\mapsto} e^x - 2e^x + e^x = 0$

$xe^x \mapsto (2e^x + xe^x) - 2(e^x + xe^x) + xe^x = 0$

$x^2e^x \mapsto (2e^x + 4xe^x + x^2e^x) - 2(2xe^x + x^2e^x) + x^2e^x = 2e^x$

$x^3e^x \mapsto (6xe^x + 6x^2e^x + x^3e^x) - 2(3x^2e^x + x^3e^x) + x^3e^x = 6xe^x$

$$\therefore \; {}_{\omega}[L]_{\alpha} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = A, \text{ REDUCING TO } \begin{array}{c} \begin{array}{cccc} \text{FREE} & \text{FREE} & & \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

α_1, α_2: FREE

$\alpha_3 = 0$

$\alpha_4 = 0$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

SO A BASIS FOR $N(A)$ IS $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

AND A BASIS FOR $C(A)$ IS $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\therefore \{e^x, xe^x\}$ IS A BASIS FOR KER L,

AND $\{2e^x, 6xe^x\}$ IS A BASIS FOR IM L.

• RANK $L = 2$, NULLITY $L = 2$: L IS NEITHER INJECTIVE NOR SURJECTIVE.

(NULLITY $\neq 0$) (RANK $\neq 4$)

(b) (TRANSLATE INTO A LINEAR SYSTEM...)

$$\begin{array}{c} \cdot\frac{1}{2} \to \\ \cdot\frac{1}{6} \to \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \\ \left[\begin{array}{cccc|c} 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{cccc} \text{FREE} & \text{FREE} & & \end{array} \\ \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

α_1, α_2: FREE

$\alpha_3 = 3/2$

$\alpha_4 = -1/3$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3/2 \\ -1/3 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore L(f(x)) = 3e^x - 2xe^x$

$$\Rightarrow f(x) = \underbrace{\left(\tfrac{3}{2}x^2e^x - \tfrac{1}{3}x^3e^x\right)}_{\text{THIS}} \;+\; \underbrace{\alpha_1 e^x + \alpha_2 xe^x}_{\text{ANYTHING IN KER } L}$$

THIS PLUS ANYTHING IN KER L

BASES α, ω

4. $V = \mathrm{SPAN}(\sin x, \sin 2x, \sin 3x) \subset C(\mathbb{R})$;

$T: V \to V, \quad f(x) \mapsto 4f(x) + f''(x)$

(a) $\sin x \overset{T}{\mapsto} 4\sin x + (-\sin x) = 3\sin x$

$\sin 2x \overset{T}{\mapsto} 4\sin 2x + (-4\sin 2x) = 0$

$\sin 3x \overset{T}{\mapsto} 4\sin 3x + (-9\sin 3x) = -5\sin 3x$

$$\therefore \; {}_{\omega}[T]_{\alpha} = \begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} \end{array} = A, \text{ REDUCING TO } \begin{array}{c} \begin{array}{ccc} & \text{FREE} & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

α_2: FREE

$\alpha_1 = 0$

$\alpha_3 = 0$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

SO A BASIS FOR $N(A)$ IS $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$,

AND A BASIS FOR $C(A)$ IS $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix} \right\}$.

$\therefore \{\sin 2x\}$ IS A BASIS FOR KER T,

AND $\{3\sin x, -5\sin 3x\}$ IS A BASIS FOR IM T.

• RANK $T = 2$, NULLITY $T = 1$: T IS NEITHER INJECTIVE NOR SURJECTIVE.

(NULLITY $\neq 0$) (RANK $\neq 3$)

(b) (TRANSLATE INTO A LINEAR SYSTEM...)

$$\begin{array}{c} \cdot\frac{1}{3} \to \\ \\ \cdot -\frac{1}{5} \to \end{array} \begin{array}{c} \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 \end{array}\right] \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccc} & \text{FREE} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & -1/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

α_2: FREE

$\alpha_1 = 1/3$

$\alpha_3 = -1/5$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ -1/5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore T(f(x)) = \sin x + \sin 3x$

$$\Rightarrow f(x) = \underbrace{\left(\tfrac{1}{3}\sin x - \tfrac{1}{5}\sin 3x\right)}_{\text{THIS}} \;+\; \underbrace{\alpha_2 \sin 2x}_{\text{ANYTHING IN KER } T}$$

THIS PLUS ANYTHING IN KER T

5. $\underset{\text{BASIS } V=(1,x,x^2)}{T: P_2(x)} \to \overset{\text{STANDARD BASIS } W}{\mathbb{R}^3}, \quad f(x) \mapsto \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}.$

(a) $1 \overset{T}{\mapsto} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad x \overset{T}{\mapsto} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad x^2 \overset{T}{\mapsto} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

$$\therefore {}_W[T]_V = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \end{array} = A, \text{ REDUCING TO } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{c} a_1, a_2, a_3 = 0 \\ \therefore \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

SO A BASIS FOR $N(A)$ IS $\{\ \}$

AND A BASIS FOR $C(A)$ IS $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\}$

$\therefore \{\ \}$ IS A BASIS FOR KER T,

AND $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\}$ IS A BASIS FOR IM T.

- RANK $T = 3$, NULLITY $T = 0$: T IS BIJECTIVE (INJECTIVE + SURJECTIVE).
  (NULLITY=0) (RANK=3)

(b) (THIS ONE IS ALREADY ONE STEP INTO COLUMN-VECTOR-NESS...)

$$\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 2 \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 7/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right] \Rightarrow \begin{array}{l} a_1 = 1 \\ a_2 = 7/2 \\ a_3 = -3/2 \end{array} \quad \therefore \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \\ -3/2 \end{bmatrix}$$

$$\therefore T(f(x)) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \Rightarrow f(x) = 1 + \tfrac{7}{2}x - \tfrac{3}{2}x^2$$

(c) [THIS IS JUST A REPHRASING OF PART (b)!
$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = T(f(x)) = \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$ JUST MEANS $f(0)=1, f(1)=3, f(2)=2$,
I.E., THE GRAPH $y=f(x)$ CONTAINS THE POINTS $(0,1), (1,3)$, AND $(2,2)$]

THERE IS EXACTLY ONE SUCH QUADRATIC: $1 + \frac{7}{2}x - \frac{3}{2}x^2$, AS IN (b).

(d)
$$\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 1 & b \\ 1 & 2 & 4 & c \end{array}\right] \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & -\frac{3}{2}a + 2b - \frac{1}{2}c \\ 0 & 0 & 1 & \frac{1}{2}a - b + \frac{1}{2}c \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ -\frac{3}{2}a + 2b - \frac{1}{2}c \\ \frac{1}{2}a - b + \frac{1}{2}c \end{bmatrix}$$

$\therefore$ THE POLYNOMIAL IS $a + \left(-\frac{3}{2}a + 2b - \frac{1}{2}c\right)x + \left(\frac{1}{2}a - b + \frac{1}{2}c\right)x^2$

(e) (JUST USE PART (d)):

(i) $(a=2, b=1, c=0)$

$2 + (-3 + 2 - 0)x + (1 - 1 + 0)x^2 = 2 - x$

(ii) $(a=10, b=4, c=-1)$

$10 + \left(-15 + 8 + \frac{1}{2}\right)x + \left(5 - 4 - \frac{1}{2}\right)x^2 = 10 - \frac{13}{2}x + \frac{1}{2}x^2$