1. [$L : \mathbb{R}^n \to \mathbb{R}^m$ LINEAR TRANSFORMATION]

LET $A = \left[L\vec{e}_1 \,\vdots\, L\vec{e}_2 \,\vdots\, \cdots \,\vdots\, L\vec{e}_n \right]$. ← ★ FORMULA FOR THE MATRIX GIVING $L : \mathbb{R}^n \to \mathbb{R}^m$!

CLAIM: $\forall\, \vec{x} \in \mathbb{R}^n$, $A\vec{x} = L\vec{x}$

PROOF: LET $\vec{x} \in \mathbb{R}^n$ BE GIVEN; THEN $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, WHERE $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

WELL, $A\vec{x} = \left[L\vec{e}_1 \,\vdots\, L\vec{e}_2 \,\vdots\, \cdots \,\vdots\, L\vec{e}_n \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\overset{\text{DEF}}{=} x_1 L\vec{e}_1 + x_2 L\vec{e}_2 + \cdots + x_n L\vec{e}_n$$

$$= L(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \quad \text{BY LINEARITY OF } L$$

$$= L\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = L\vec{x} \quad \blacksquare$$

★ NET EFFECT: EVERY L.T. FROM $\mathbb{R}^n \to \mathbb{R}^m$ IS GIVEN BY A MATRIX!

2. [V : n-DIMENSIONAL VECTOR SPACE, WITH ORDERED BASIS $\mathcal{V}$
W : m-DIMENSIONAL VECTOR SPACE, WITH ORDERED BASIS $\mathcal{W}$
$L : V \to W$ LINEAR TRANSFORMATION] ★

(a) GIVEN THE ABOVE INGREDIENTS, WE CAN FORM A LINEAR TRANSFORMATION ${}_{\mathcal{W}}[L]_{\mathcal{V}} : \mathbb{R}^n \to \mathbb{R}^m$ AS FOLLOWS:

$$ {}_{\mathcal{W}}[L]_{\mathcal{V}} = [\mathcal{W}]^{-1}\, L\, [\mathcal{V}] $$

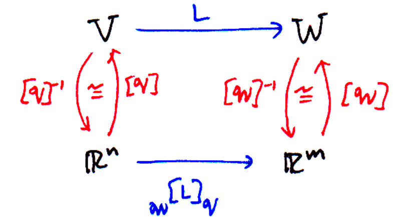
$\mathbb{R}^m \leftarrow W \leftarrow V \leftarrow \mathbb{R}^n$

BEING A L.T. : $\mathbb{R}^n \to \mathbb{R}^m$, ${}_{\mathcal{W}}[L]_{\mathcal{V}}$ IS GIVEN BY A MATRIX.

(b) THE F.T.L.T. SAYS THAT GIVEN THE SITUATION ★ ABOVE, L CAN BE REPRESENTED BY A MATRIX ${}_{\mathcal{W}}[L]_{\mathcal{V}}$:

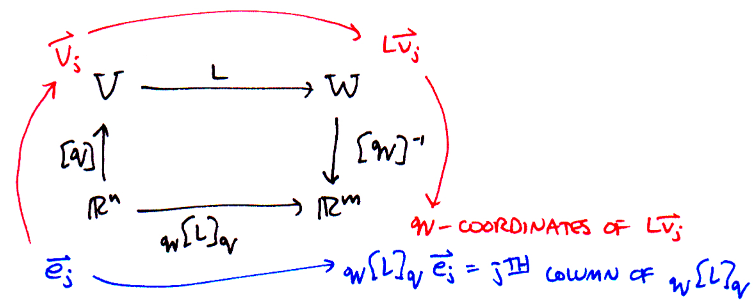
$$ L = [\mathcal{W}]\left({}_{\mathcal{W}}[L]_{\mathcal{V}}\right)[\mathcal{V}]^{-1} $$

(c) THE WORKINGS OF THE F.T.L.T. ARE ILLUSTRATED BY THE FOLLOWING COMMUTATIVE DIAGRAM:



$$ \left(\begin{array}{l} \Rightarrow L = [\mathcal{W}]\left({}_{\mathcal{W}}[L]_{\mathcal{V}}\right)[\mathcal{V}]^{-1} : V \to W \\ \text{AND } {}_{\mathcal{W}}[L]_{\mathcal{V}} = [\mathcal{W}]^{-1} L [\mathcal{V}] : \mathbb{R}^n \to \mathbb{R}^m \end{array} \right) $$

(d) WRITING OUR BASIS $\mathcal{V} = (\vec{v}_1, \ldots, \vec{v}_n)$, WE CAN ANALYZE THE ABOVE DIAGRAM TO DETERMINE HOW TO COMPUTE ${}_{\mathcal{W}}[L]_{\mathcal{V}}$:



BY COMMUTATIVITY, WE FIND THAT:

THE j^{TH} COLUMN OF ${}_{\mathcal{W}}[L]_{\mathcal{V}}$ = THE $\mathcal{W}$-COORDINATES OF $L\vec{v}_j$

(e) ${}_{\mathcal{W}}[L]_{\mathcal{V}}$ IS NOT INTRINSIC TO L — IT DEPENDS ON OUR CHOICES OF BASES $\mathcal{V}, \mathcal{W}$!

3. (a) $L:\mathbb{R}^3\to\mathbb{R}^2$, $\begin{bmatrix}x\\y\\z\end{bmatrix}\mapsto\begin{bmatrix}3x-y+z\\z+y-2x\end{bmatrix}$

$$L\vec{e}_1=L\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix};\ L\vec{e}_2=L\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix};\ L\vec{e}_3=L\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$\therefore$ L IS GIVEN BY THE MATRIX $\begin{bmatrix}3&-1&1\\-2&1&1\end{bmatrix}$.

(b) $L:\mathbb{R}^4\to\mathbb{R}^3$, $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\mapsto\begin{bmatrix}z+x-2y\\10y+2x-z+3w\\w+7x-y\end{bmatrix}$

$$L\vec{e}_1=L\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\7\end{bmatrix};\ L\vec{e}_2=L\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-2\\10\\-1\end{bmatrix};\ L\vec{e}_3=L\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix};\ L\vec{e}_4=L\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\3\\1\end{bmatrix}$$

$\therefore$ L IS GIVEN BY THE MATRIX $\begin{bmatrix}1&-2&1&0\\2&10&-1&3\\7&-1&0&1\end{bmatrix}$.

(c) $L:\mathbb{R}^2\to\mathbb{R}^3$, $\begin{bmatrix}x\\y\end{bmatrix}\mapsto\begin{bmatrix}x+y\\y-x\\3x\end{bmatrix}$

$$L\vec{e}_1=L\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\3\end{bmatrix};\quad L\vec{e}_2=L\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$\therefore$ L IS GIVEN BY THE MATRIX $\begin{bmatrix}1&1\\-1&1\\3&0\end{bmatrix}$.

(d) $L:\mathbb{R}^3\to\mathbb{R}^4$, $\begin{bmatrix}x\\y\\z\end{bmatrix}\mapsto\begin{bmatrix}x+y+z\\x+2y+3z\\x+4y+9z\\z-y\end{bmatrix}$

$$L\vec{e}_1=L\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\1\\0\end{bmatrix};\quad L\vec{e}_2=L\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\\4\\-1\end{bmatrix};\quad L\vec{e}_3=L\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\3\\9\\1\end{bmatrix}$$

$\therefore$ L IS GIVEN BY THE MATRIX $\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\\0&-1&1\end{bmatrix}$.

4. [D: THE L.T. OF DIFFERENTIATION (WITH RESPECT TO x)]

(a) [$V=P_3(x)$, WITH BASIS $(1,x,x^2,x^3)$; $W=P_2(x)$, WITH BASIS $(1,x,x^2)$]

• 1^{ST} COL. OF ${}_W[D]_V$ = W-COORD'S OF $D\vec{v}_1$:

$$\underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}\leftrightarrow\underset{\vec{v}_1}{1}\overset{D}{\rightsquigarrow}\underset{D\vec{v}_1}{0}=0\cdot1+0\cdot x+0\cdot x^2\leftrightarrow\underset{\text{W-COORD'S OF }D\vec{v}_1}{\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

• 2^{ND} COL. OF ${}_W[D]_V$ = W-COORD'S OF $D\vec{v}_2$:

$$\underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}\leftrightarrow\underset{\vec{v}_2}{x}\overset{D}{\rightsquigarrow}\underset{D\vec{v}_2}{1}=1\cdot1+0\cdot x+0\cdot x^2\leftrightarrow\underset{\text{W-COORD'S OF }D\vec{v}_2}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$$

• 3^{RD} COL. OF ${}_W[D]_V$ = W-COORD'S OF $D\vec{v}_3$:

$$\underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\leftrightarrow\underset{\vec{v}_3}{x^2}\overset{D}{\rightsquigarrow}\underset{D\vec{v}_3}{2x}=0\cdot1+2\cdot x+0\cdot x^2\leftrightarrow\underset{\text{W-COORD'S OF }D\vec{v}_3}{\begin{bmatrix}0\\2\\0\end{bmatrix}}$$

• 4^{TH} COL. OF ${}_W[D]_V$ = W-COORD'S OF $D\vec{v}_4$:

$$\underset{\vec{e}_4}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}\leftrightarrow\underset{\vec{v}_4}{x^3}\overset{D}{\rightsquigarrow}\underset{D\vec{v}_4}{3x^2}=0\cdot1+0\cdot x+3\cdot x^2\leftrightarrow\underset{\text{W-COORD'S OF }D\vec{v}_4}{\begin{bmatrix}0\\0\\3\end{bmatrix}}$$

$$\therefore\ {}_W[D]_V=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}.$$

(b) $\begin{bmatrix} V = P_3(x), \text{ with basis } \mathcal{V} = (x, x^2, x^3, 1) \\ W = P_2(x), \text{ with basis } \mathcal{W} = (1, 2x, 3x^2) \end{bmatrix}$

- 1st col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_1}{x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_1}{1} = 1\cdot 1 + 0\cdot(2x) + 0\cdot(3x^2) \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$$

- 2nd col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_2}{x^2} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_2}{2x} = 0\cdot 1 + 1\cdot(2x) + 0\cdot(3x^2) \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}$$

- 3rd col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_3}{x^3} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_3}{3x^2} = 0\cdot 1 + 0\cdot(2x) + 1\cdot(3x^2) \leftrightarrow \underset{\mathcal{W}\text{-coords of } D\vec{v}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

- 4th col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_4}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \leftrightarrow \underset{\vec{v}_4}{1} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_4}{0} = 0\cdot 1 + 0\cdot(2x) + 0\cdot(3x^2) \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_4}{\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

$$\therefore\ {}_{\mathcal{W}}[D]_{\mathcal{V}} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}.$$

(c) $[V, W = \text{span}(\underbrace{\sin x, \cos x, \sin 2x, \cos 2x}_{\mathcal{V},\mathcal{W}}) \subset C(\mathbb{R})]$

- 1st col of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_1}{\sin x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_1}{\cos x} = 0\cdot\sin x + 1\cdot\cos x + 0\cdot\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-coords of } D\vec{v}_1}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}$$

- 2nd col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_2}{\cos x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_2}{-\sin x} = -1\cdot\sin x + 0\cdot\cos x + 0\cdot\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_2}{\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}}$$

- 3rd col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_3}{\sin 2x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_3}{2\cos 2x} = 0\cdot\sin x + 0\cdot\cos x + 0\cdot\sin 2x + 2\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_3}{\begin{bmatrix}0\\0\\0\\2\end{bmatrix}}$$

- 4th col. of ${}_{\mathcal{W}}[D]_{\mathcal{V}}$:

$$\underset{\vec{e}_4}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \leftrightarrow \underset{\vec{v}_4}{\cos 2x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_4}{-2\sin 2x} = 0\cdot\sin x + 0\cdot\cos x + (-2)\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-coord's of } D\vec{v}_4}{\begin{bmatrix}0\\0\\-2\\0\end{bmatrix}}$$

$$\therefore\ {}_{\mathcal{W}}[D]_{\mathcal{V}} = \begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-2\\0&0&2&0\end{bmatrix}.$$

(d) $[V, W = \text{SPAN}(\underbrace{e^x, xe^x, e^{-x}, xe^{-x}}_{\mathcal{V},\mathcal{W}}) \subset C(\mathbb{R})]$

- $\underline{1^{ST}\text{ COL. OF }{}_{\mathcal{W}}[D]_{\mathcal{V}}}$:

$$\underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_1}{e^x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_1}{e^x} = 1\cdot e^x + 0\cdot(xe^x) + 0\cdot e^{-x} + 0\cdot(xe^{-x}) \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }D\vec{v}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}$$

- $\underline{2^{ND}\text{ COL. OF }{}_{\mathcal{W}}[D]_{\mathcal{V}}}$:

$$\underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_2}{xe^x} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_2}{e^x + xe^x} = 1\cdot e^x + 1\cdot xe^x + 0\cdot e^{-x} + 0\cdot xe^{-x} \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }D\vec{v}_2}{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}$$

- $\underline{3^{RD}\text{ COL. OF }{}_{\mathcal{W}}[D]_{\mathcal{V}}}$:

$$\underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_3}{e^{-x}} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_3}{-e^{-x}} = 0\cdot e^x + 0\cdot xe^x + (-1)e^{-x} + 0\cdot xe^{-x} \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }D\vec{v}_3}{\begin{bmatrix}0\\0\\-1\\0\end{bmatrix}}$$

- $\underline{4^{TH}\text{ COL. OF }{}_{\mathcal{W}}[D]_{\mathcal{V}}}$:

$$\underset{\vec{e}_4}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \leftrightarrow \underset{\vec{v}_4}{xe^{-x}} \overset{D}{\rightsquigarrow} \underset{D\vec{v}_4}{e^{-x} - xe^{-x}} = 0\cdot e^x + 0\cdot xe^{-x} + 1\cdot e^{-x} + (-1)\cdot xe^{-x} \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }D\vec{v}_4}{\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}}$$

$$\therefore\ {}_{\mathcal{W}}[D]_{\mathcal{V}} = \begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&-1&1\\0&0&0&-1\end{bmatrix}.$$

5. $[L: f(x) \mapsto f(x) + f''(x)]$

(c) $V, W = \text{SPAN}(\underbrace{\sin x, \cos x, \sin 2x, \cos 2x}_{\mathcal{V},\mathcal{W}}) \subset C(\mathbb{R})$

- $\underline{1^{ST}\text{ COL}}$:

$$\underset{\vec{e}_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_1}{\sin x} \overset{L}{\rightsquigarrow} \underset{L\vec{v}_1}{\sin x + (-\sin x)} = 0$$
$$= 0\cdot\sin x + 0\cdot\cos x + 0\cdot\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }L\vec{v}_1}{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}$$

- $\underline{2^{ND}\text{ COL}}$:

$$\underset{\vec{e}_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_2}{\cos x} \overset{L}{\rightsquigarrow} \underset{L\vec{v}_2}{\cos x + (-\cos x)} = 0$$
$$= 0\cdot\sin x + 0\cdot\cos x + 0\cdot\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }L\vec{v}_2}{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}$$

- $\underline{3^{RD}\text{ COL}}$:

$$\underset{\vec{e}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}} \leftrightarrow \underset{\vec{v}_3}{\sin 2x} \overset{L}{\rightsquigarrow} \underset{L\vec{v}_3}{\sin 2x - 4\sin 2x} = -3\sin 2x$$
$$= 0\cdot\sin x + 0\cdot\cos x + (-3)\sin 2x + 0\cdot\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }L\vec{v}_3}{\begin{bmatrix}0\\0\\-3\\0\end{bmatrix}}$$

- $\underline{4^{TH}\text{ COL}}$:

$$\underset{\vec{e}_4}{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}} \leftrightarrow \underset{\vec{v}_4}{\cos 2x} \overset{L}{\rightsquigarrow} \underset{L\vec{v}_4}{\cos 2x - 4\cos 2x} = -3\cos 2x$$
$$= 0\cdot\sin x + 0\cdot\cos x + 0\cdot\sin 2x + (-3)\cos 2x \leftrightarrow \underset{\mathcal{W}\text{-COORD'S OF }L\vec{v}_4}{\begin{bmatrix}0\\0\\0\\-3\end{bmatrix}}$$

$$\therefore\ {}_{\mathcal{W}}[L]_{\mathcal{V}} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-3&0\\0&0&0&-3\end{bmatrix}.$$

$[L : f(x) \mapsto f(x) + f''(x)]$

(d) $V, W = \text{SPAN}\underbrace{(e^x, xe^x, e^{-x}, xe^{-x})}_{\mathcal{V}, \mathcal{W}} \subset C(\mathbb{R})$

- 1ST COL: $\underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{\vec{e}_1} \leftrightarrow \underbrace{e^x}_{\vec{v}_1} \overset{L}{\rightsquigarrow} \underbrace{e^x + e^x}_{L\vec{v}_1} = 2e^x$

  $= 2e^x + 0 \cdot xe^x + 0 \cdot e^{-x} + 0 \cdot xe^{-x} \leftrightarrow \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$ $\mathcal{W}$-COORD.'S OF $L\vec{v}_1$

- 2ND COL: $\underbrace{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\vec{e}_2} \leftrightarrow \underbrace{xe^x}_{\vec{v}_2} \overset{L}{\rightsquigarrow} xe^x + (2e^x + xe^x) = \underbrace{2e^x + 2xe^x}_{L\vec{v}_2}$

  $= 2e^x + 2 \cdot xe^x + 0 \cdot e^{-x} + 0 \cdot xe^{-x} \leftrightarrow \begin{bmatrix}2\\2\\0\\0\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_2$

- 3RD COL: $\underbrace{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}_{\vec{e}_3} \leftrightarrow \underbrace{e^{-x}}_{\vec{v}_3} \overset{L}{\rightsquigarrow} \underbrace{e^{-x} + e^{-x}}_{L\vec{v}_3} = 2e^{-x}$

  $= 0 \cdot e^x + 0 \cdot xe^x + 2 \cdot e^{-x} + 0 \cdot xe^{-x} \leftrightarrow \begin{bmatrix}0\\0\\2\\0\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_3$

- 4TH COL: $\underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{\vec{e}_4} \longleftrightarrow \underbrace{xe^{-x}}_{\vec{v}_4} \overset{L}{\rightsquigarrow} xe^{-x} + (-2e^{-x} + xe^{-x}) = \underbrace{-2e^{-x} + 2xe^{-x}}_{L\vec{v}_4}$

  $= 0 \cdot e^x + 0 \cdot xe^x + (-2)\, e^{-x} + 2 \cdot xe^{-x} \leftrightarrow \begin{bmatrix}0\\0\\-2\\2\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_4$

$$\therefore\ {}_{\mathcal{W}}[L]_{\mathcal{V}} = \begin{bmatrix}2 & 2 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & -2\\ 0 & 0 & 0 & 2\end{bmatrix}.$$

6. $[L : P_3(x) \to P_3(x),\ f(x) \mapsto xf'(x) - 2f(x)\ ;\ \mathcal{V} = \mathcal{W} = (1, x, x^2, x^3)]$

- 1ST COL: $\underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{\vec{e}_1} \leftrightarrow \underbrace{1}_{\vec{v}_1} \overset{L}{\rightsquigarrow} 1 \cdot 0 - 2 \cdot 1 = -2$ $L\vec{v}_1$

  $= -2 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \leftrightarrow \begin{bmatrix}-2\\0\\0\\0\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_1$

- 2ND COL: $\underbrace{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\vec{e}_2} \leftrightarrow \underbrace{x}_{\vec{v}_2} \overset{L}{\rightsquigarrow} x \cdot 1 - 2 \cdot x = -x$ $L\vec{v}_2$

  $= 0 \cdot 1 + (-1)x + 0 \cdot x^2 + 0 \cdot x^3 \leftrightarrow \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_2$

- 3RD COL: $\underbrace{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}_{\vec{e}_3} \leftrightarrow \underbrace{x^2}_{\vec{v}_3} \overset{L}{\rightsquigarrow} x \cdot (2x) - 2 \cdot x^2 = 0$ $L\vec{v}_3$

  $= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \longleftrightarrow \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_3$

- 4TH COL: $\underbrace{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}}_{\vec{e}_4} \leftrightarrow \underbrace{x^3}_{\vec{v}_4} \overset{L}{\rightsquigarrow} x \cdot (3x^2) - 2 \cdot x^3 = x^3$ $L\vec{v}_4$

  $= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3 \longleftrightarrow \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ $\mathcal{W}$-COORD'S OF $L\vec{v}_4$

$$\therefore\ {}_{\mathcal{W}}[L]_{\mathcal{V}} = \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$