[V, W: FINITE-DIMENSIONAL VECTOR SPACES; $L: V \to W$ L.T.]

1. CLAIM: IF $\{\vec{a}_1, \dots, \vec{a}_m\}$ SPANS V, THEN $\{L\vec{a}_1, \dots, L\vec{a}_m\}$ SPANS IM L.

PROOF: SUPPOSE THAT $\{\vec{a}_1, \dots, \vec{a}_m\}$ SPANS V,

I.E., $\forall\, \vec{v} \in V$, $\exists$ SCALARS $q_1, \dots, q_m$ (!) WITH $\vec{v} = q_1\vec{a}_1 + \dots + q_m\vec{a}_m$

[NEED TO SHOW: $\{L\vec{a}_1, \dots, L\vec{a}_m\}$ SPANS IM L

I.E., $\forall\, \vec{w} \in$ IM L, $\exists$ SCALARS $q_1, \dots, q_m$ WITH $\vec{w} = q_1 L\vec{a}_1 + \dots + q_m L\vec{a}_m$

MIGHT AS WELL — HERE AGAIN, THE SAME COEFF'S WILL WORK FOR BOTH!

LET $\vec{w} \in$ IM L BE GIVEN; THEN BY DEFINITION OF IM L, $\vec{w} = L\vec{v}$ FOR SOME $\vec{v} \in V$.

[NEED TO FIND SCALARS $q_1, \dots, q_m$ ... USE HYPOTHESIS?]

BY HYPOTHESIS, $\exists$ SCALARS $q_1, \dots, q_m$ WITH $\vec{v} = q_1\vec{a}_1 + \dots + q_m\vec{a}_m$

[SCALARS FOUND... JUST SHOW THAT $\vec{w} = q_1 L\vec{a}_1 + \dots + q_m L\vec{a}_m$]

TAKE SUCH SCALARS $q_1, \dots, q_m$. THEN

$$\begin{aligned} q_1 L\vec{a}_1 + q_2 L\vec{a}_2 + \dots + q_m L\vec{a}_m &= L(q_1\vec{a}_1 + q_2\vec{a}_2 + \dots + q_m\vec{a}_m) && \text{BY LINEARITY OF } L \\ &= L(\vec{v}) = \vec{w} && \text{FROM ABOVE.} \end{aligned}$$

SO, BY DEFINITION, $\{L\vec{a}_1, \dots, L\vec{a}_m\}$ SPANS IM L ■

2. • RANK $L \overset{\text{DEF}}{=}$ DIM (IM L)

($0 \le$ RANK $L \le$ DIM W BECAUSE IM L IS A SUBSPACE OF W)

• NULLITY $L \overset{\text{DEF}}{=}$ DIM (KER L)

($0 \le$ NULLITY $L \le$ DIM V BECAUSE KER L IS A SUBSPACE OF V)

[V, W: FINITE-DIMENSIONAL V.S.'S, $L: V \to W$ LINEAR TRANSFORMATION]

3. (a) KER $L \subset V$ IS A FINITE-DIMENSIONAL VECTOR SPACE, SO IT HAS SOME BASIS $\{\vec{k}_1, \dots, \vec{k}_n\}$. n IS THE SIZE OF THIS BASIS FOR KER L, SO BY DEFINITION, $n =$ NULLITY L.

(b) BECAUSE $\{\vec{k}_1, \dots, \vec{k}_n\}$ IS LINEARLY INDEPENDENT, WE CAN EXTEND IT TO A BASIS $\{\vec{k}_1, \dots, \vec{k}_n; \vec{v}_1, \dots, \vec{v}_r\}$ FOR V. THIS BASIS FOR V HAS $n+r$ VECTORS, SO $n + r =$ DIM V.

(c) $\{L\vec{k}_1, \dots, L\vec{k}_n; L\vec{v}_1, \dots, L\vec{v}_r\}$ SPANS IM L BY PROBLEM #1; BUT SINCE $\vec{k}_1, \dots, \vec{k}_n \in$ KER L, WE HAVE $L\vec{k}_1 = \vec{0}, \dots, L\vec{k}_n = \vec{0}$, SO THESE CAN BE REMOVED WITHOUT CHANGING THE SPAN OF THIS COLLECTION, BECAUSE THEY'RE L.C.'S OF THE REST.
$\therefore$ $\{L\vec{v}_1, \dots, L\vec{v}_r\}$ SPANS IM L.

(I.E., $q_1 L\vec{v}_1 + \dots + q_r L\vec{v}_r = \vec{0} \Rightarrow q_1, \dots, q_r = 0$)

(d) CLAIM: $\{L\vec{v}_1, \dots, L\vec{v}_r\}$ IS LINEARLY INDEPENDENT.

PROOF: SUPPOSE THAT $q_1 L\vec{v}_1 + \dots + q_r L\vec{v}_r = \vec{0}$

THEN BY LINEARITY, $L(q_1\vec{v}_1 + \dots + q_r\vec{v}_r) = \vec{0}$,

SO BY DEFINITION OF KER L, $q_1\vec{v}_1 + \dots + q_r\vec{v}_r \in$ KER L.

SINCE $\{\vec{k}_1, \dots, \vec{k}_n\}$ IS A BASIS FOR KER L, IT SPANS KER L;

$\therefore$ $\exists$ SCALARS $\beta_1, \dots, \beta_n$ WITH $q_1\vec{v}_1 + \dots + q_r\vec{v}_r = \beta_1\vec{k}_1 + \dots + \beta_n\vec{k}_n$

BUT THEN $\beta_1\vec{k}_1 + \dots + \beta_n\vec{k}_n - q_1\vec{v}_1 - \dots - q_r\vec{v}_r = \vec{0}$, LINEAR RELATION!

SO SINCE $\{\vec{k}_1, \dots, \vec{k}_n; \vec{v}_1, \dots, \vec{v}_r\}$ IS A BASIS FOR V (AND THUS IS LINEARLY INDEPENDENT!), THIS FORCES

$\beta_1, \dots, \beta_n, q_1, \dots, q_r = 0$, SO $q_1, \dots, q_r = 0$

$\therefore$ BY DEFINITION, $\{L\vec{v}_1, \dots, L\vec{v}_r\}$ IS LINEARLY INDEPENDENT. ■

BECAUSE THIS COLLECTION SPANS IM L, IT IS THUS A BASIS FOR IM L, SO BY DEFINITION, $r =$ RANK L.

(e) [FROM (6), $\underbrace{n}_{\substack{\text{NULLITY } L,\\ \text{BY (a)}}} + \underbrace{r}_{\substack{\text{RANK } L,\\ \text{BY (d)}}} = \dim V$]

THE RANK + NULLITY THEOREM: IF V IS A FINITE-DIMENSIONAL VECTOR SPACE AND $L: V \to W$ IS A LINEAR TRANSFORMATION, THEN RANK L + NULLITY L = DIM V.

↳ A SORT OF CONSERVATION LAW FOR L.T.'S — OF THE DIM V WORTH OF DOMAIN, L KILLS NULLITY L WORTH OF IT, LEAVING RANK L WORTH OF IT FOR THE IMAGE.

4. [A: $m \times n$ MATRIX, GIVING A L.T. $A: \mathbb{R}^n \to \mathbb{R}^m$]

RECALL THAT IF WE ROW-REDUCE A:

- A BASIS FOR $C(A)$ IS GIVEN BY THE COLUMNS OF A THAT GIVE PIVOTS, SO

  RANK A = DIM $C(A)$ = # PIVOT COLUMNS.

- A BASIS FOR $N(A)$ IS OBTAINED FROM THE FREE VARIABLES' CONTRIBUTIONS TO THE SOLUTION OF THE ASSOCIATED HOMOGENEOUS SYSTEM, SO

  NULLITY A = DIM $N(A)$ = # FREE COLUMNS.

THUS, SINCE DIM $\mathbb{R}^n = n$ = # OF COLUMNS, WE SEE THAT IN THE CASE OF A MATRIX, THE RANK + NULLITY THEOREM IS DEMONSTRATED BY THE SPLIT OF THE COLUMNS INTO PIVOT COLUMNS AND NON-PIVOT (FREE) COLUMNS!

[$L: \underset{n\text{-DIM'L}}{V} \to \underset{m\text{-DIM'L}}{W}$ LINEAR TRANSFORMATION]

5. (a) NULLITY $L = 0 \Rightarrow \ker L = \{\vec{0}\}$, SO L IS INJECTIVE.

↳ THE ONLY V.S. WITH DIMENSION 0 IS $\{\vec{0}\}$

(b) NULLITY $L = n \Rightarrow \ker L = V$, I.E., $\forall \vec{v} \in V$, $L\vec{v} = \vec{0}$ — L IS THE ZERO MAP.

↳ $\ker L$ IS A SUBSPACE OF V THAT HERE HAS THE SAME DIMENSION AS V; [NOT] EXTENDING A BASIS FOR $\ker L$ TO A BASIS FOR V SHOWS US THAT IT IS ALSO A BASIS FOR V, SO $\ker L = V$.

(c) RANK $L = 0 \Rightarrow \operatorname{IM} L = \{0\}$, SO L IS AGAIN THE ZERO MAP.

↳ BY REASONING AS IN PART (a)

(d) RANK $L = m \Rightarrow \operatorname{IM} L = W$, SO L IS SURJECTIVE.

↳ BY REASONING AS IN PART (b)

6. - IF $n < m$, THEN RANK $L = n -$ NULLITY $L \le n < m$.
  SINCE RANK $L < m$, $\dim(\operatorname{IM} L) < m = \dim \mathbb{R}^m$;
  SO IM L CANNOT BE ALL OF $\mathbb{R}^m$;
  I.E., L CANNOT BE SURJECTIVE.

- IF $m < n$, THEN NULLITY $L = n -$ RANK $L \ge n - m > 0$;
  THUS $\dim(\ker L) > 0$, SO $\ker L \ne \{0\}$,
  THEREFORE L CANNOT BE INJECTIVE.

7. (a)

$$\begin{array}{c}\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}\\ \underbrace{\begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & -4 \\ 1 & 0 & 7 \\ -2 & 1 & -7 \end{bmatrix}}\end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1&=0\\ x_2&=0\\ x_3&=0\end{aligned} \;\Rightarrow\; \vec{x}=\begin{bmatrix} x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

(i) 3 VECTORS IN $\mathbb{R}^4$, SO $A:\mathbb{R}^3\to\mathbb{R}^4$.

(ii) $N(A)$ HAS BASIS $\{\}$; $C(A)$ HAS BASIS $\left\{\begin{bmatrix}1\\0\\1\\-2\end{bmatrix},\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix},\begin{bmatrix}-2\\-4\\7\\-7\end{bmatrix}\right\}$.

↳ ZERO-DIMENSIONAL — NO FREE VARIABLES!

(iii) RANK IS 3; NULLITY IS 0; $0+3=3$ ✓

(iv) A IS INJECTIVE, BECAUSE IT HAS NULLITY 0;
A IS NOT SURJECTIVE BECAUSE ITS RANK IS $3<4=\dim\mathbb{R}^4$.

(b)

$$\begin{array}{c}\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}\\ \underbrace{\begin{bmatrix} 1 & -2 & -4 & -10 \\ 3 & -5 & -10 & -27 \\ -2 & 6 & 13 & 29 \end{bmatrix}}\end{array} \rightsquigarrow \begin{array}{c}\begin{array}{cccc} & & & \text{FREE} \end{array}\\ \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}\end{array} \qquad \begin{aligned} x_1&=4x_4\\ x_2&=3x_4\\ x_3&=-3x_4\end{aligned} \;\Rightarrow\; \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}4x_4\\3x_4\\-3x_4\\x_4\end{bmatrix}=x_4\begin{bmatrix}4\\3\\-3\\1\end{bmatrix}$$

(i) 4 VECTORS IN $\mathbb{R}^3$, SO $A:\mathbb{R}^4\to\mathbb{R}^3$

(ii) $N(A)$ HAS BASIS $\left\{\begin{bmatrix}4\\3\\-3\\1\end{bmatrix}\right\}$; $C(A)$ HAS BASIS $\left\{\begin{bmatrix}1\\3\\-2\end{bmatrix},\begin{bmatrix}-2\\-5\\6\end{bmatrix},\begin{bmatrix}-4\\-10\\13\end{bmatrix}\right\}$

(iii) RANK IS 3; NULLITY IS 1; $3+1=4$ ✓

(iv) A IS NOT INJECTIVE, BECAUSE IT HAS NULLITY $1>0$;
A IS SURJECTIVE BECAUSE ITS RANK IS $3=\dim\mathbb{R}^3$.

(c)

$$\begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\ \underbrace{\begin{bmatrix} -1 & -2 & -1 & -3 & 8 \\ 1 & 3 & -2 & 2 & -2 \\ 0 & 1 & -5 & -3 & 12 \\ -1 & -3 & 0 & -4 & 8 \end{bmatrix}}\end{array} \rightsquigarrow \begin{array}{c}\begin{array}{ccccc} & & & \text{FREE} & \text{FREE} \end{array}\\ \begin{bmatrix} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array} \qquad \begin{aligned} x_1&=2x_4-x_5\\ x_2&=-2x_4+3x_5\\ x_3&=-x_4+3x_5\end{aligned}$$

$$\Rightarrow \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2x_4-x_5\\-2x_4+3x_5\\-x_4+3x_5\\x_4\\x_5\end{bmatrix}=x_4\begin{bmatrix}2\\-2\\-1\\1\\0\end{bmatrix}+x_5\begin{bmatrix}-1\\3\\3\\0\\1\end{bmatrix}$$

(i) 5 VECTORS IN $\mathbb{R}^4$,
SO $A:\mathbb{R}^5\to\mathbb{R}^4$

(ii) $N(A)$ HAS BASIS $\left\{\begin{bmatrix}2\\-2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\3\\3\\0\\1\end{bmatrix}\right\}$; $C(A)$ HAS BASIS $\left\{\begin{bmatrix}-1\\1\\0\\-1\end{bmatrix},\begin{bmatrix}-2\\3\\1\\-3\end{bmatrix},\begin{bmatrix}-1\\-2\\-5\\0\end{bmatrix}\right\}$

(iii) RANK IS 3; NULLITY IS 2; $3+2=5$ ✓

(iv) A IS NOT INJECTIVE, BECAUSE IT HAS NULLITY $2>0$;
A IS NOT SURJECTIVE EITHER, BECAUSE IT HAS RANK $3<4=\dim\mathbb{R}^4$.