1. $[\mathcal{C} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$: ORDERED COLLECTION IN A VECTOR SPACE $V]$

(a) $[\mathcal{C}] : \mathbb{R}^n \longrightarrow V$ IS THE LINEAR COMBINATION FUNCTION, DEFINED AS FOLLOWS: (DOMAIN $\mathbb{R}^n$, CODOMAIN V)

$$\underset{\substack{\uparrow \\ \text{COLLECTION}}}{[\mathcal{C}]}\underset{\substack{\uparrow \\ \text{COEFFICIENT} \\ \text{VECTOR}}}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underbrace{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n}_{\text{CORRESPONDING L.C. OF } \mathcal{C}}$$

(b) $[\mathcal{C}]$ IS A LINEAR TRANSFORMATION FROM $\mathbb{R}^n$ TO V:

⓪ (SHOW $[\mathcal{C}]\vec{0} = \vec{0}$)

$$[\mathcal{C}]\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0 \cdot \vec{v}_n = \vec{0} + \cdots + \vec{0} = \vec{0} \quad ✓$$

① (SHOW THAT $\forall\, \vec{x} \in \mathbb{R}^n$ AND $\alpha \in \mathbb{R}$, $[\mathcal{C}](\alpha\vec{x}) = \alpha[\mathcal{C}]\vec{x}$)

LET $\vec{x} \in \mathbb{R}^n$ AND $\alpha \in \mathbb{R}$ BE GIVEN; THEN $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, WHERE $x_1, \ldots, x_n \in \mathbb{R}$.

NOW, $$[\mathcal{C}](\alpha\vec{x}) = [\mathcal{C}]\left(\alpha\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = [\mathcal{C}]\begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$
$$= (\alpha x_1)\vec{v}_1 + (\alpha x_2)\vec{v}_2 + \cdots + (\alpha x_n)\vec{v}_n$$
$$= \alpha(x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n)$$
$$= \alpha[\mathcal{C}]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \alpha[\mathcal{C}]\vec{x} \quad ✓$$

② (SHOW THAT $\forall\, \vec{x}, \vec{y} \in \mathbb{R}^n$, $[\mathcal{C}](\vec{x}+\vec{y}) = [\mathcal{C}]\vec{x} + [\mathcal{C}]\vec{y}$)

LET $\vec{x}, \vec{y} \in \mathbb{R}^n$ BE GIVEN; THEN $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ AND $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, WHERE $x_i, y_i \in \mathbb{R}$.

NOW, $$[\mathcal{C}](\vec{x}+\vec{y}) = [\mathcal{C}]\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = [\mathcal{C}]\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$
$$= (x_1+y_1)\vec{v}_1 + (x_2+y_2)\vec{v}_2 + \cdots + (x_n+y_n)\vec{v}_n$$
$$= (x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n) + (y_1\vec{v}_1 + y_2\vec{v}_2 + \cdots + y_n\vec{v}_n)$$
$$= [\mathcal{C}]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [\mathcal{C}]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [\mathcal{C}]\vec{x} + [\mathcal{C}]\vec{y} \quad ✓$$

BY ⓪, ①, AND ②, $[\mathcal{C}] : \mathbb{R}^n \to V$ IS A LINEAR TRANSFORMATION ■

2. $[\mathcal{C} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$: ORDERED COLLECTION IN A VECTOR SPACE $V]$

(a) $\operatorname{IM}[\mathcal{C}]$ IS JUST THE SPAN OF $\mathcal{C}$!

WHY? $$\operatorname{IM}[\mathcal{C}] \overset{\text{DEF}}{=} \{[\mathcal{C}]\vec{x} : \vec{x} \in \mathbb{R}^n\}$$
$$= \left\{[\mathcal{C}]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R}\right\}$$
$$= \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n : x_1, \ldots, x_n \in \mathbb{R}\}$$
$$\overset{\text{DEF}}{=} \operatorname{SPAN} \mathcal{C}$$ (THE SET OF ALL L.C.'S OF $\mathcal{C}$!)

(b) $\operatorname{KER}[\mathcal{C}]$ GIVES US THE COEFFICIENT VECTORS FOR ALL POSSIBLE LINEAR RELATIONS ON $\mathcal{C}$.

WHY? $$\operatorname{KER}[\mathcal{C}] \overset{\text{DEF}}{=} \{\vec{x} \in \mathbb{R}^n : [\mathcal{C}]\vec{x} = \vec{0}\}$$
$$= \left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R} \text{ AND } [\mathcal{C}]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}\right\}$$
$$= \left\{\underset{\uparrow}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} : x_1, \ldots, x_n \in \mathbb{R} \text{ AND } \underset{\uparrow}{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}}\right\}$$

THE SET OF ALL COEFFICIENT VECTORS THAT GIVE LINEAR RELATIONS ON $\mathcal{C}$.

3. [$\mathcal{C}=(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n)$: ORDERED COLLECTION IN A VECTOR SPACE V]

(a) IF $\mathcal{C}$ IS L.I. IN V, THEN $[\mathcal{C}]$ IS INJECTIVE,
BECAUSE KER $[\mathcal{C}]$ WILL BE $\{\vec{0}\}$.

WHY? AS NOTED IN 2(b), THE VECTORS OF KER $[\mathcal{C}]$ CORRESPOND TO THE LINEAR RELATIONS ON $\mathcal{C}$; IF $\mathcal{C}$ IS L.I., THEN THE ONLY L.R. ON $\mathcal{C}$ IS THE TRIVIAL ONE, SO KER $[\mathcal{C}]=\{\vec{0}\}$, WHICH FROM THE PREVIOUS PROBLEM SET WE KNOW MEANS THAT $[\mathcal{C}]$ IS INJECTIVE.

ALTERNATIVE EXPLANATION (NOT AS GOOD OF PRACTICE WITH USING OUR NEW TERMINOLOGY): IF $\mathcal{C}$ IS L.I. IN V, THEN EACH VECTOR OF V CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$ IN AT MOST ONE WAY, SO $[\mathcal{C}]$ CAN HIT EACH $\vec{v}\in V$ AT MOST ONCE — I.E., $[\mathcal{C}]$ IS INJECTIVE.

(b) IF $\mathcal{C}$ SPANS V, THEN $[\mathcal{C}]$ IS SURJECTIVE,
BECAUSE IM $[\mathcal{C}]$ WILL BE ALL OF V.

WHY? AS NOTED IN 2(a), IM $[\mathcal{C}]$ = SPAN $\mathcal{C}$; IF $\mathcal{C}$ SPANS V, THEN IM $[\mathcal{C}]$ = SPAN $\mathcal{C}=V$, SO BY DEFINITION, $[\mathcal{C}]$ IS SURJECTIVE.

ALTERNATIVE EXPLANATION: IF $\mathcal{C}$ SPANS V, THEN EACH VECTOR OF V CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$ IN AT LEAST ONE WAY, SO $[\mathcal{C}]$ HITS EACH $\vec{v}\in V$ AT LEAST ONCE — I.E., $[\mathcal{C}]$ IS SURJECTIVE.

(c) IF $\mathcal{C}$ IS A BASIS FOR V, THEN $[\mathcal{C}]$ IS BIJECTIVE.

WHY? IF $\mathcal{C}$ IS A BASIS FOR V, THEN $\mathcal{C}$ IS L.I. IN V AND SPANS V, SO BY (a) AND (b), $[\mathcal{C}]$ IS BOTH INJECTIVE AND SURJECTIVE — I.E., $[\mathcal{C}]$ IS BIJECTIVE.

ALTERNATIVE EXPLANATION: IF $\mathcal{C}$ IS A BASIS FOR V, THEN EACH VECTOR OF V CAN BE WRITTEN AS A L.C. OF $\mathcal{C}$ IN EXACTLY ONE WAY, SO $[\mathcal{C}]$ PAIRS UP EACH $\vec{x}\in\mathbb{R}^n$ WITH ONE $\vec{v}\in V$ AND VICE-VERSA — I.E., $[\mathcal{C}]$ IS BIJECTIVE.

* A BIJECTIVE LINEAR TRANSFORMATION IS, BY DEFINITION, AN ISOMORPHISM.

4. [$\mathcal{C}=(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_n)$: ORDERED COLLECTION IN $\mathbb{R}^m$; $[\mathcal{C}]\rightsquigarrow$ MATRIX A]

E.G., $\mathcal{C}=\left(\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}4\\1\\5\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix},\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right)$ IN $\mathbb{R}^3$: $[\mathcal{C}]\rightsquigarrow\begin{bmatrix}1&4&0&-1\\0&1&2&1\\2&5&0&-1\end{bmatrix}=A$

(a) A HAS $\underset{\text{ROWS: SIZE OF COLUMN VECTORS}}{m}$ ROWS AND $\underset{\text{COLS: \# OF COLUMN VECTORS}}{n}$ COLUMNS;

E.G. AS ABOVE: A IS A 3×4 MATRIX.

THUS, IT IS AN m BY n MATRIX.
DIMENSIONS OF THE MATRIX: ROWS × COLUMNS

(b) IF $\vec{x}\in\mathbb{R}^n$, $A\vec{x}\overset{\text{DEF}}{=}[\mathcal{C}]\vec{x}$ IS A LINEAR COMBINATION OF $\mathcal{C}$,
I.E., A LINEAR COMBINATION OF THE COLUMNS OF A
(WITH COEFFICIENTS GIVEN BY THE ENTRIES OF $\vec{x}$).

E.G. AS ABOVE: $\begin{bmatrix}1&4&0&-1\\0&1&2&1\\2&5&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\2\\-1\end{bmatrix}=1\begin{bmatrix}1\\0\\2\end{bmatrix}+0\begin{bmatrix}4\\1\\5\end{bmatrix}+2\begin{bmatrix}0\\2\\0\end{bmatrix}-1\begin{bmatrix}-1\\1\\-1\end{bmatrix}=\begin{bmatrix}2\\3\\3\end{bmatrix}$

(c) JUST AS WITH $[\mathcal{C}]$, $A:\overset{\text{DOMAIN}}{\mathbb{R}^n}\longrightarrow\overset{\text{CODOMAIN}}{\mathbb{R}^m}$

COEFFICIENTS IN $\mathbb{R}^n\longmapsto$ L.C. IN $\mathbb{R}^m$

E.G. AS ABOVE, $A:\mathbb{R}^4\longrightarrow\mathbb{R}^3$.

$\{A: m\times n$ MATRIX ARISING FROM $\mathcal{C}=(\vec{v}_1,\ldots,\vec{v}_n)$ IN $\mathbb{R}^m\}$

5. $N(A)=\{\vec{x}\in\mathbb{R}^n : A\vec{x}=\vec{0}\}$ (I.E., THE KERNEL OF THE L.T. A)

- $N(A)$ IS A SUBSPACE OF $\mathbb{R}^n$ (THE DOMAIN OF THE L.T. A)
- WE CAN FIND A BASIS FOR $N(A)$ SIMPLY BY SOLVING THE HOMOGENEOUS SYSTEM $A\vec{x}=\vec{0}$ AND SPLITTING OUR SOLUTION AS THE SPAN OF ITS FREE VARIABLES' CONTRIBUTIONS.

E.G. AS ABOVE, TO FIND A BASIS FOR $N(A)$, SOLVE:

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \hline 1 & 4 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 2 & 5 & 0 & -1 \end{array} \rightsquigarrow \begin{bmatrix} \textcircled{1} & 0 & 0 & \overset{\text{FREE}}{1/3} \\ 0 & \textcircled{1} & 0 & -1/3 \\ 0 & 0 & \textcircled{1} & 2/3 \end{bmatrix}$$

a_4 : FREE
$a_1=-\frac{1}{3}a_4$
$a_2=\frac{1}{3}a_4$
$a_3=-\frac{2}{3}a_4$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}a_4 \\ \frac{1}{3}a_4 \\ -\frac{2}{3}a_4 \\ a_4 \end{bmatrix} = a_4\begin{bmatrix} -1/3 \\ 1/3 \\ -2/3 \\ 1 \end{bmatrix}$$

SO $\left\{\begin{bmatrix} -1/3 \\ 1/3 \\ -2/3 \\ 1 \end{bmatrix}\right\}$ IS A BASIS FOR $N(A)$.

6. $C(A)=\{A\vec{x} : \vec{x}\in\mathbb{R}^n\}$ (I.E., THE IMAGE OF THE L.T. A)

- $C(A)$ IS A SUBSPACE OF $\mathbb{R}^m$ (THE CODOMAIN OF THE L.T. A)
- WE CAN FIND A BASIS FOR $C(A)$ BY REDUCING A AND TAKING THE COLUMNS (OF OUR ORIGINAL MATRIX A!) THAT GAVE PIVOTS (JUST AS WE DO WHEN FINDING THE BASIS OF A SPAN).

E.G. AS ABOVE, A BASIS FOR $C(A)$ IS $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}4\\1\\5\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$.

7. (COMPUTING THE RESULT OF A MATRIX ACTING ON A COLUMN VECTOR IS JUST COMPUTING A LINEAR COMBINATION!)

(a) $$\begin{bmatrix} 2 & 1 \\ -3 & 0 \\ -1 & 5 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -3 \\ -1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 5 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ -3 \\ -11 \\ -6 \end{bmatrix}}$$

(b) $$\begin{bmatrix} 3 & 0 & 1 & 4 \\ -2 & -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \underline{\begin{bmatrix} 14 \\ 15 \end{bmatrix}}$$

(c) $$\begin{bmatrix} 4 & 1 & 3 & 0 \\ 2 & -2 & 0 & 5 \\ 1 & 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}}$$

8. Ae_j IS JUST THE j^{TH} COLUMN OF A.

↳ ALL 0'S EXCEPT FOR A 1 IN THE j^{TH} ENTRY

(b) $A=\begin{bmatrix} 3 & 0 & 1 & 4 \\ -2 & -2 & 1 & 5 \end{bmatrix}$; $\vec{x}=\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}=0\vec{e}_1+1\vec{e}_2+2\vec{e}_3+3\vec{e}_4$

$$\begin{aligned} A\vec{x} &= A(0\vec{e}_1+1\vec{e}_2+2\vec{e}_3+3\vec{e}_4) \\ &= 0\cdot A\vec{e}_1 + 1\cdot A\vec{e}_2 + 2\cdot A\vec{e}_3 + 3\cdot A\vec{e}_4 \quad \text{BY LINEARITY OF } A \\ &= 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \end{bmatrix}, \end{aligned}$$

JUST AS IN 7(b).

9. $\{A : m \times n$ MATRIX$\}$

RECALL THAT: • $N(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$ ⟷ L.R.'S ON THE COLUMNS OF A

• $C(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ = SPAN OF THE COLUMNS OF A

(a) $A\vec{x} = \vec{0}$ ONLY FOR $\vec{x} = \vec{0}$ TELLS US THAT $N(A) = \{\vec{0}\}$, SO THE ONLY L.R. ON THE COLUMNS OF A IS THE TRIVIAL ONE — I.E., THE COLUMNS OF A ARE LINEARLY INDEPENDENT.

(b) $A\vec{x} = \vec{0}$ FOR SOME $\vec{x} \neq \vec{0}$ TELLS US THAT $N(A) \neq \{\vec{0}\}$, SO THERE IS A NONTRIVIAL L.R. ON THE COLUMNS OF A — I.E., THE COLUMNS OF A ARE LINEARLY DEPENDENT.

(c) $A\vec{x} = \vec{b}$ HAS A SOLUTION TELLS US THAT $\vec{b} \in C(A)$ — I.E., THAT $\vec{b}$ IS IN THE SPAN OF THE COLUMNS OF A.

(d) $A\vec{x} = \vec{b}$ HAS NO SOLUTIONS TELLS US THAT $\vec{b} \notin C(A)$ — I.E., THAT $\vec{b}$ IS NOT IN THE SPAN OF THE COLUMNS OF A.

(e) FOR EACH $\vec{b} \in \mathbb{R}^m$, THE EQUATION $A\vec{x} = \vec{b}$ HAS A UNIQUE SOLUTION TELLS US THAT EVERY $\vec{b} \in \mathbb{R}^m$ CAN BE WRITTEN AS A L.C. OF THE COLUMNS OF A IN EXACTLY ONE WAY, SO THE COLUMNS OF A ARE A BASIS FOR $\mathbb{R}^m$. (ALTERNATIVELY, THIS TELLS US THAT WHEN A IS REDUCED, WE OBTAIN A PIVOT IN EVERY ROW + COLUMN... SO, AGAIN, THE COLUMNS OF A FORM A BASIS FOR $\mathbb{R}^m$).