1. IF V AND W ARE VECTOR SPACES, A <u>LINEAR TRANSFORMATION</u> FROM V TO W IS A <u>FUNCTION</u> $L: V \to W$ (V IS THE <u>DOMAIN</u> OF L, AND W IS ITS <u>CODOMAIN</u>.) POSSESSING THE PROPERTY OF <u>LINEARITY</u> (I.E., RESPECTING L.C.'S):

$$\forall \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ AND SCALARS } \alpha_1, \alpha_2, \ldots, \alpha_n,$$
$$L(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n) = \alpha_1 L(\vec{v}_1) + \alpha_2 L(\vec{v}_2) + \cdots + \alpha_n L(\vec{v}_n)$$

* THIS SHOULD LOOK FAMILIAR! IT'S THE SAME CONDITION PLACED ON ISOMORPHISMS (THE SAME COMMUTATIVE DIAGRAM CAN BE USED TO EXPLAIN THE MEANING OF THIS CONDITION). THE DIFFERENCE IS THAT A LINEAR TRANSFORMATION IS <u>NOT</u> REQUIRED TO BE BIJECTIVE (WHEREAS AN ISOMORPHISM IS).

AS WITH SUBSPACES, WE CAN CHECK THIS CONDITION ON ALL L.C.'S IN THREE STEPS:

⓪ $L(\vec{0}_V) = \vec{0}_W$

① $\forall \vec{v} \in V$ AND SCALAR α, $L(\alpha\vec{v}) = \alpha L(\vec{v})$

② $\forall \vec{v}_1, \vec{v}_2 \in V$, $L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$

(IF SO, $L(\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n) \overset{②}{=} L(\alpha_1\vec{v}_1) + \cdots + L(\alpha_n\vec{v}_n)$

$\overset{①}{=} \alpha_1 L(\vec{v}_1) + \cdots + \alpha_n L(\vec{v}_n)$, AS REQUIRED OF A L.T.)

2. $L: V \to W$ LINEAR TRANSFORMATION

(a) THE <u>IMAGE OF L</u> IS DEFINED BY:

$$\text{IM } L = \{ L(\vec{v}) : \vec{v} \in V \}$$

I.E., THE OUTPUT VALUES THAT RESULT FROM ALL POSSIBLE INPUT VALUES

THE IMAGE OF L IS THE LINEAR ALGEBRA TERM FOR THE <u>RANGE</u> (SET OF OUTPUT VALUES) OF THE FUNCTION L.

(b) <u>CLAIM</u>: IM L IS A SUBSPACE OF W

<u>PROOF</u>: ⓪ (NEED TO SHOW THAT $\vec{0}_W \in$ IM L, I.E., $\vec{0}_W = L(\vec{v})$ FOR SOME $\vec{v} \in V$)

$\vec{0}_V \in V$, AND BY PROPERTY ⓪ OF L.T.'S, $L(\vec{0}_V) = \vec{0}_W$, SO BY DEFINITION OF IM L, $\vec{0}_W \in$ IM L ✓

① (NEED TO SHOW THAT $\forall \vec{w}_1 \in$ IM L AND SCALAR α, $\alpha\vec{w}_1 \in$ IM L)

LET $\vec{w}_1 \in$ IM L AND SCALAR α BE GIVEN.

BY DEFINITION OF IM L, $\vec{w}_1 = L(\vec{v}_1)$ FOR SOME $\vec{v}_1 \in V$.

[NEED TO SHOW $\alpha\vec{w}_1 \in$ IM L, I.E., $\alpha\vec{w}_1 = L(\vec{v})$ FOR SOME $\vec{v} \in V$. WHAT $\vec{v}$ TO TAKE? WELL, $\alpha\vec{w}_1 = \alpha L(\vec{v}_1) = L(\alpha\vec{v}_1)$ BY ① FOR L.T.'S...)

LET $\vec{v} = \alpha\vec{v}_1 \in V$.

THEN $L(\vec{v}) = L(\alpha\vec{v}_1) = \alpha L(\vec{v}_1)$ BY PROPERTY ① OF L.T.'S

$= \alpha\vec{w}_1$,

SO BY DEFINITION OF IM L, $\alpha\vec{w}_1 \in$ IM L ✓

② (NEED TO SHOW THAT $\forall\, \vec{w}_1, \vec{w}_2 \in \mathrm{IM}\, L$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{IM}\, L$)

LET $\vec{w}_1, \vec{w}_2 \in \mathrm{IM}\, L$ BE GIVEN.

BY DEFINITION OF IM L, $\vec{w}_1 = L(\vec{v}_1)$ FOR SOME $\vec{v}_1 \in V$
AND $\vec{w}_2 = L(\vec{v}_2)$ FOR SOME $\vec{v}_2 \in V$.

[NEED TO SHOW $\vec{w}_1 + \vec{w}_2 \in \mathrm{IM}\, L$, I.E,
$\vec{w}_1 + \vec{w}_2 = L(\vec{v})$ FOR SOME $\vec{v} \in V$. WHAT $\vec{v}$ TO TAKE?
WELL, $\vec{w}_1 + \vec{w}_2 = L(\vec{v}_1) + L(\vec{v}_2) = L(\vec{v}_1 + \vec{v}_2)$ BY ② FOR L.T.'s...)

LET $\vec{v} = \vec{v}_1 + \vec{v}_2 \in V$.

THEN $L(\vec{v}) = L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$ BY LINEARITY OF L
$= \vec{w}_1 + \vec{w}_2$,

SO BY DEFINITION OF IM L, $\vec{w}_1 + \vec{w}_2 \in \mathrm{IM}\, L$ ✓

BY ⓪, ①, AND ②, IM L IS A SUBSPACE OF W ■

(c) IF IM $L = W$, THEN BY DEFINITION, L IS SURJECTIVE (ONTO),
I.E., EVERY $\vec{w} \in W$ IS "HIT" BY SOME $\vec{v} \in V$.

3. $L : V \to W$ LINEAR TRANSFORMATION

(a) THE KERNEL OF L IS DEFINED BY:

$$\ker L = \{ \vec{v} \in V : L(\vec{v}) = \vec{0}_W \}$$

ALL VECTORS OF V THAT L MAPS TO $\vec{0}$

THIS SIMPLY TELLS US WHICH VECTORS GET MAPPED TO THE ZERO VECTOR BY L.

(b) CLAIM: KER L IS A SUBSPACE OF V.

PROOF: ⓪ (NEED TO SHOW $\vec{0}_V \in \ker L$, I.E., $L(\vec{0}_V) = \vec{0}_W$)

$L(\vec{0}_V) = \vec{0}_W$ BY LINEARITY OF L,

SO BY DEFINITION OF KER L, $\vec{0}_V \in \ker L$. ✓

① (NEED TO SHOW $\forall\, \vec{v}_1 \in \ker L$ AND SCALAR q, $q\vec{v}_1 \in \ker L$)

LET $\vec{v}_1 \in \ker L$ AND SCALAR q BE GIVEN.

THEN BY DEFINITION OF KER L, $L(\vec{v}_1) = \vec{0}_W$.

[NEED TO SHOW $q\vec{v}_1 \in \ker L$, I.E., $L(q\vec{v}_1) = \vec{0}$ ...]

WELL, $L(q\vec{v}_1) = qL(\vec{v}_1)$ BY LINEARITY OF L
$= q \cdot \vec{0}_W = \vec{0}_W$,

SO BY DEFINITION OF KER L, $q\vec{v}_1 \in \ker L$. ✓

② (NEED TO SHOW $\forall\, \vec{v}_1, \vec{v}_2 \in \ker L$, $\vec{v}_1 + \vec{v}_2 \in \ker L$)

LET $\vec{v}_1, \vec{v}_2 \in \ker L$ BE GIVEN.

BY DEFINITION OF KER L, $L(\vec{v}_1) = \vec{0}_W$ AND $L(\vec{v}_2) = \vec{0}_W$

[NEED TO SHOW $\vec{v}_1 + \vec{v}_2 \in \ker L$, I.E., $L(\vec{v}_1 + \vec{v}_2) = \vec{0}_W$.]

WELL, $L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$ BY LINEARITY OF L
$= \vec{0}_W + \vec{0}_W = \vec{0}_W$,

SO BY DEFINITION OF KER L, $\vec{v}_1 + \vec{v}_2 \in \ker L$. ✓

BY ⓪, ①, AND ②, KER L IS A SUBSPACE OF V ■

(c) IF $\ker L = \{\vec{0}\}$, THEN L IS INJECTIVE, I.E., ONE-TO-ONE
(SEE PROBLEM 5).

4. $L: V \to W$ AND $K: W \to U$ LINEAR TRANSFORMATIONS

(a) $K \circ L : V \to U$ IS DEFINED BY $\vec{v} \mapsto K(L(\vec{v}))$;
ITS DOMAIN IS V, AND ITS CODOMAIN IS U.

* REMEMBER THAT COMPOSITIONS ACT RIGHT-TO-LEFT!
$(K \circ L)(\vec{v}) = K(\underline{L(\vec{v})})$
L FIRST, THEN K

(b) CLAIM: IF $L: V \to W$ AND $K: W \to U$ ARE L.T.'s, THEN $K \circ L : V \to U$ IS A LINEAR TRANSFORMATION

PROOF: SUPPOSE $L: V \to W$ AND $K: W \to U$ ARE L.T.'S.
[BECAUSE LINEARITY HAS SUCH A NICE FUNCTIONAL FORM, WE OFTEN WILL JUST USE IT LIKE A RULE RATHER THAN UNRAVELING IT FULLY INTO SYMBOLIC LOGIC — WE JUST KNOW THAT K + L RESPECT L.C.'S]

TO SHOW $K \circ L$ IS A L.T:

⓪ (NEED TO SHOW THAT $(K \circ L)(\vec{0}_V) = \vec{0}_U$ — JUST COMPUTE!)

$$(K \circ L)(\vec{0}_V) \overset{DEF}{=} K(L(\vec{0}_V)) = K(\vec{0}_W) \text{ BY LINEARITY OF } L$$
$$= \vec{0}_U \text{ BY LINEARITY OF } K \checkmark$$

① (NEED TO SHOW $\forall \vec{v} \in V$ AND SCALAR α, $(K \circ L)(\alpha \vec{v}) = \alpha (K \circ L)(\vec{v})$)
LET $\vec{v} \in V$ AND SCALAR α BE GIVEN.

THEN $(K \circ L)(\alpha \vec{v}) \overset{DEF}{=} K(L(\alpha \vec{v})) = K(\alpha L(\vec{v}))$ BY LINEARITY OF L
$= \alpha K(L(\vec{v}))$ BY LINEARITY OF K
$\overset{DEF}{=} \alpha (K \circ L)(\vec{v})$ ✓

② (NEED TO SHOW $\forall \vec{v}_1, \vec{v}_2 \in V$, $(K \circ L)(\vec{v}_1 + \vec{v}_2) = (K \circ L)(\vec{v}_1) + (K \circ L)(\vec{v}_2)$)
LET $\vec{v}_1, \vec{v}_2 \in V$ BE GIVEN.

THEN $(K \circ L)(\vec{v}_1 + \vec{v}_2) \overset{DEF}{=} K(L(\vec{v}_1 + \vec{v}_2))$
$= K(L(\vec{v}_1) + L(\vec{v}_2))$ BY LINEARITY OF L
$= K(L(\vec{v}_1)) + K(L(\vec{v}_2))$ BY LINEARITY OF K
$\overset{DEF}{=} (K \circ L)(\vec{v}_1) + (K \circ L)(\vec{v}_2)$ ✓

BY ⓪, ①, AND ②, $K \circ L$ IS A LINEAR TRANSFORMATION ■

5. $L: V \to W$ LINEAR TRANSFORMATION

MANY ⟺ PROOFS REQUIRE US TO DO TWO SUB-PROOFS (⟹ AND ⟸); THIS ONE, HOWEVER, IS SIMPLE ENOUGH TO PULL OFF WITH A CHAIN OF ⟺'s!

(a) CLAIM: $L(\vec{x}) = L(\vec{y}) \iff \vec{x} - \vec{y} \in \ker L$

PROOF: $L(\vec{x}) = L(\vec{y})$
$\iff L(\vec{x}) - L(\vec{y}) = \vec{0}$
$\iff L(\vec{x} - \vec{y}) = \vec{0}$ (BY LINEARITY OF L)
$\iff \vec{x} - \vec{y} \in \ker L$ (BY DEFINITION OF KER L) ■

(b) CLAIM: IF $L(\vec{x}_0) = \vec{a}$, THEN:
$L(\vec{x}) = \vec{a} \iff \exists \vec{v} \in \ker L$ WITH $\vec{x} = \vec{x}_0 + \vec{v}$. *

PROOF: SUPPOSE $L(\vec{x}_0) = \vec{a}$. [NEED TO SHOW *]

• $L(\vec{x}) = \vec{a} \Rightarrow \exists \vec{v} \in \ker L$ WITH $\vec{x} = \vec{x}_0 + \vec{v}$:

SUPPOSE $L(\vec{x}) = \vec{a}$.
$\therefore L(\vec{x}) = L(\vec{x}_0)$

[NEED TO FIND SOME $\vec{v} \in \ker L$... WHERE CAN WE GO WITH THIS? WELL, $\vec{a} = L(\vec{x}_0)$ BY HYPOTHESIS.]

SO BY PART (a), $\vec{x} - \vec{x}_0 \in \ker L$ AHA!
TAKE $\vec{v} = \vec{x} - \vec{x}_0 \in \ker L$. THEN $\vec{x}_0 + \vec{v} = \vec{x}_0 + (\vec{x} - \vec{x}_0) = \vec{x}$,
SO $\exists \vec{v} \in \ker L$ WITH $\vec{x} = \vec{x}_0 + \vec{v}$ ✓

• [$\exists \vec{v} \in \ker L$ WITH $\vec{x} = \vec{x}_0 + \vec{v}$] $\Rightarrow L(\vec{x}) = \vec{a}$:

[SHOW $L(\vec{x}) = \vec{a}$]

SUPPOSE $\exists \vec{v} \in \ker L$ WITH $\vec{x} = \vec{x}_0 + \vec{v}$.
TAKING SUCH A $\vec{v}$, WE THEN HAVE

$L(\vec{x}) = L(\vec{x}_0 + \vec{v}) = L(\vec{x}_0) + L(\vec{v})$ BY LINEARITY OF L
$= \vec{a} + \vec{0}_W$ BECAUSE $L(\vec{x}_0) = \vec{a}$ AND $\vec{v} \in \ker L$
$= \vec{a}$ ✓

■

6. $L: V \to W$ AND $K: W \to U$ LINEAR TRANSFORMATIONS.

CLAIM: $\vec{x} \in \ker(K \circ L) \iff L(\vec{x}) \in \ker K$ ← THIS COULD BE UNRAVELED AND PROVEN AS $\Rightarrow + \Leftarrow$, BUT IT'S SIMPLE ENOUGH TO PROVE VIA A CHAIN OF $\Leftrightarrow$'s

PROOF: $\vec{x} \in \ker(K \circ L)$

$\iff (K \circ L)(\vec{x}) = \vec{0}_U$ BY DEFINITION OF $\ker(K \circ L)$

$\iff K(\underline{L(\vec{x})}) = \vec{0}_U$

K OF THIS IS ZERO...

$\iff L(\vec{x}) \in \ker K$, BY DEFINITION OF $\ker K$. ■

7. (a) $L: \mathbb{R}^3 \to \mathbb{R}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto xy + z$ [IS THIS A L.T.? ⓪, ①, ②]

⓪ (SHOW THAT $L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = 0$)

$L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = 0 \cdot 0 + 0 = 0$ ✓

① (SHOW THAT $\forall \vec{v} \in \mathbb{R}^3$ AND $\alpha \in \mathbb{R}$, $L(\alpha\vec{v}) = \alpha L(\vec{v})$)

LET $\vec{v} \in \mathbb{R}^3$ AND $\alpha \in \mathbb{R}$ BE GIVEN; THEN $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ FOR SOME $x, y, z \in \mathbb{R}$

SO $\alpha\vec{v} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}$, AND THUS $L(\alpha\vec{v}) = (\alpha x)(\alpha y) + \alpha z = \alpha^2 xy + \alpha z$

NOT LOOKING GOOD!

WHILE $\alpha L(\vec{v}) = \alpha L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \alpha(xy + z) = \alpha xy + \alpha z$

TO JUSTIFY THAT THIS IS BAD, FIND α, x, y, z SO THAT THIS FAILS — E.G., $\alpha = 2$, $x = y = z = 1$

CONDITION ① IS FALSE — E.G.,

TAKING $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ AND $\alpha = 2$,

WE HAVE $L(\alpha\vec{v}) = L\left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right) = 4 + 2 = 6$,

WHILE $\alpha L(\vec{v}) = 2L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = 2(1+1) = 4$

$4 \neq 6$, SO ① FAILS!

$\therefore$ L IS NOT A LINEAR TRANSFORMATION.

(b) $L: \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3x - y + z \\ z + y - 2x \end{bmatrix}$ [IS THIS A L.T.? ⓪, ①, ②]

⓪ (SHOW THAT $L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

$L\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 - 0 + 0 \\ 0 + 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

① (SHOW THAT $\forall \vec{v} \in \mathbb{R}^3$ AND $\alpha \in \mathbb{R}$, $L(\alpha\vec{v}) = \alpha L(\vec{v})$)

LET $\vec{v} \in \mathbb{R}^3$ AND $\alpha \in \mathbb{R}$ BE GIVEN; THEN $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ FOR SOME $x, y, z \in \mathbb{R}$.

$\alpha\vec{v} = \alpha\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}$, SO $L(\alpha\vec{v}) = \begin{bmatrix} 3(\alpha x) - (\alpha y) + (\alpha z) \\ (\alpha z) + (\alpha y) - 2(\alpha x) \end{bmatrix}$

$= \begin{bmatrix} 3\alpha x - \alpha y + \alpha z \\ \alpha z + \alpha y - 2\alpha x \end{bmatrix}$

$= \alpha \begin{bmatrix} 3x - y + z \\ z + y - 2x \end{bmatrix} = \alpha L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \alpha L(\vec{v})$ ✓

② (SHOW THAT $\forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$, $L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$)

LET $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ BE GIVEN. THEN $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ AND $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

FOR SOME $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$, SO $L(\vec{v}_1 + \vec{v}_2) = \begin{bmatrix} 3(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) \\ (z_1 + z_2) + (y_1 + y_2) - 2(x_1 + x_2) \end{bmatrix}$

$= \begin{bmatrix} (3x_1 - y_1 + z_1) + (3x_2 - y_2 + z_2) \\ (z_1 + y_1 - 2x_1) + (z_2 + y_2 - 2x_2) \end{bmatrix}$

$= \begin{bmatrix} 3x_1 - y_1 + z_1 \\ z_1 + y_1 - 2x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 - y_2 + z_2 \\ z_2 + y_2 - 2x_2 \end{bmatrix}$

$= L(\vec{v}_1) + L(\vec{v}_2)$ ✓

BY ⓪, ①, AND ②, L IS A LINEAR TRANSFORMATION.

(c) $L: C(\mathbb{R}) \to \mathbb{R}^2$, $f \mapsto \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}$ [IS THIS A L.T.? ⓪,①,②]

⓪ (SHOW THAT $L(\vec{0}_{C(\mathbb{R})}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

$\vec{0}_{C(\mathbb{R})}$ IS THE ZERO FUNCTION $f: x \mapsto 0$,

SO $L(\vec{0}_{C(\mathbb{R})}) = L(f) = \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

① (SHOW THAT $\forall f \in C(\mathbb{R})$ AND $\alpha \in \mathbb{R}$, $L(\alpha f) = \alpha L(f)$)

LET $f \in C(\mathbb{R})$ AND $\alpha \in \mathbb{R}$ BE GIVEN.

THEN $L(\alpha f) = \begin{bmatrix} (\alpha f)(-1) \\ (\alpha f)(1) \end{bmatrix} = \begin{bmatrix} \alpha f(-1) \\ \alpha f(1) \end{bmatrix} = \alpha \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} = \alpha L(f)$ ✓

② (SHOW THAT $\forall f_1, f_2 \in C(\mathbb{R})$, $L(f_1 + f_2) = L(f_1) + L(f_2)$)

LET $f_1, f_2 \in C(\mathbb{R})$ BE GIVEN.

THEN $L(f_1+f_2) = \begin{bmatrix} (f_1+f_2)(-1) \\ (f_1+f_2)(1) \end{bmatrix}$

$= \begin{bmatrix} f_1(-1) + f_2(-1) \\ f_1(1) + f_2(1) \end{bmatrix}$

$= \begin{bmatrix} f_1(-1) \\ f_1(1) \end{bmatrix} + \begin{bmatrix} f_2(-1) \\ f_2(1) \end{bmatrix} = L(f_1) + L(f_2)$ ✓

BY ⓪, ①, AND ②, L IS A LINEAR TRANSFORMATION.

(d) $L: C(\mathbb{R}) \to \mathbb{R}$, $f \mapsto f(0) + 1$ [IS THIS A L.T.? ⓪,①,②]

⓪ (SHOW THAT $L(\vec{0}_{C(\mathbb{R})}) = 0$)

$\vec{0}_{C(\mathbb{R})}$ IS THE FUNCTION $f: x \mapsto 0$,

SO $L(\vec{0}_{C(\mathbb{R})}) = L(f) = f(0) + 1 = 0 + 1 = 1 \neq 0$

$\therefore$ L IS NOT A LINEAR TRANSFORMATION.

(e) $L: C(\mathbb{R}) \to \mathbb{R}$, $f \mapsto 2f(0) - f(1)$ [IS THIS A L.T.? ⓪,①,②]

⓪ (SHOW THAT $L(\vec{0}_{C(\mathbb{R})}) = 0$)

$\vec{0}_{C(\mathbb{R})}$ IS THE FUNCTION $f: x \mapsto 0$,

SO $L(\vec{0}_{C(\mathbb{R})}) = L(f) = 2f(0) - f(1) = 2 \cdot 0 - 0 = 0$ ✓

① (SHOW THAT $\forall f \in C(\mathbb{R})$ AND $\alpha \in \mathbb{R}$, $L(\alpha f) = \alpha L(f)$)

LET $f \in C(\mathbb{R})$ AND $\alpha \in \mathbb{R}$ BE GIVEN.

THEN $L(\alpha f) = 2(\alpha f)(0) - (\alpha f)(1) = 2\alpha f(0) - \alpha f(1)$

$= \alpha(2f(0) - f(1))$

$= \alpha L(f)$ ✓

② (SHOW THAT $\forall f_1, f_2 \in C(\mathbb{R})$, $L(f_1+f_2) = L(f_1) + L(f_2)$)

LET $f_1, f_2 \in C(\mathbb{R})$ BE GIVEN.

THEN $L(f_1+f_2) = 2(f_1+f_2)(0) - (f_1+f_2)(1)$

$= 2[f_1(0) + f_2(0)] - [f_1(1) + f_2(1)]$

$= 2f_1(0) - f_1(1) + 2f_2(0) - f_2(1)$

$= L(f_1) + L(f_2)$ ✓

BY ⓪, ①, AND ②, L IS A LINEAR TRANSFORMATION.

(f) $L: \mathbb{R}^2 \to \mathbb{R}[x]$, $\begin{bmatrix} A \\ B \end{bmatrix} \mapsto A + Bx + (A+B)x^2$ [IS THIS A L.T.? ⓪①②]

⓪ (SHOW THAT $L(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0$)

$L(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0 + 0x + (0+0)x^2 = 0$ ✓

① (SHOW THAT $\forall \vec{v} \in \mathbb{R}^2$ AND $\alpha \in \mathbb{R}$, $L(\alpha\vec{v}) = \alpha L(\vec{v})$)

LET $\vec{v} \in \mathbb{R}^2$ AND $\alpha \in \mathbb{R}$ BE GIVEN; THEN $\vec{v} = \begin{bmatrix} A \\ B \end{bmatrix}$ FOR SOME $A, B \in \mathbb{R}$.

$\alpha\vec{v} = \alpha\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \alpha A \\ \alpha B \end{bmatrix}$, SO $L(\alpha\vec{v}) = (\alpha A) + (\alpha B)x + (\alpha A + \alpha B)x^2$

$= \alpha(A + Bx + (A+B)x^2)$

$= \alpha L(\begin{bmatrix} A \\ B \end{bmatrix}) = \alpha L(\vec{v})$ ✓

② (SHOW THAT $\forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, $L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2)$)

LET $\vec{v}_1, \vec{v}_2 \in \mathbb{R}$ BE GIVEN;

THEN $\vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ AND $\vec{v}_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ FOR SOME $A_1, B_1, A_2, B_2 \in \mathbb{R}$.

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} A_1 + A_2 \\ B_1 + B_2 \end{bmatrix}$,

SO $L(\vec{v}_1 + \vec{v}_2) = (A_1 + A_2) + (B_1 + B_2)x + ((A_1 + A_2) + (B_1 + B_2))x^2$

$= [A_1 + B_1 x + (A_1 + B_1)x^2] + [A_2 + B_2 x + (A_2 + B_2)x^2]$

$= L(\vec{v}_1) + L(\vec{v}_2)$ ✓

BY ⓪, ①, AND ②, L IS A LINEAR TRANSFORMATION.

8. [TO FIND THE KERNEL OF A L.T., JUST SET UP + SOLVE $L(\vec{v}) = \vec{0}$!]

(b) $L(\vec{v}) = \vec{0}$ HERE MEANS $L(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

I.E., $\begin{bmatrix} 3x - y + z \\ z + y - 2x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

OR, $x\begin{bmatrix} 3 \\ -2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 3 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{matrix} & & \text{FREE} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix} \end{array}$$

z: FREE
$x = -2z$
$y = -5z$

THE SOLUTION VECTORS THUS ARE $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ -5z \\ z \end{bmatrix}$ FOR $z \in \mathbb{R}$,

SO $\ker(L)$ IS JUST $\left\{ \begin{bmatrix} -2z \\ -5z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$

(f) $L(\vec{v}) = 0$ HERE MEANS $L(\begin{bmatrix} A \\ B \end{bmatrix}) = 0$

I.E., $A + Bx + (A+B)x^2 = 0$

REMEMBER THAT WE'RE NOT SOLVING FOR x, BUT A, B, C!

SO $\underset{\text{CONSTANTS}}{A = 0}$, $\underset{x\text{'s}}{B = 0}$, $\underset{x^2\text{'s}}{A + B = 0}$

THUS THE ONLY SOLUTION IS $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

SO $\ker(L) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

9. $L: \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x-y+z \end{bmatrix}$; $K: \mathbb{R}^2 \to \mathbb{R}$, $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto u-3v$

(a) $K \circ L: \mathbb{R}^3 \to \mathbb{R}$, SO $\mathbb{R}^3$ IS THE DOMAIN OF $K \circ L$ AND $\mathbb{R}$ IS THE CODOMAIN OF $K \circ L$.

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "L"] \arrow[rr, bend right, dotted, "K \circ L"'] & \mathbb{R}^2 \arrow[r, "K"] & \mathbb{R} \end{tikzcd}$$

(b) $(K \circ L)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = K\left(L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) = K\left(\begin{bmatrix} 2x+y \\ x-y+z \end{bmatrix}\right) = (2x+y) - 3(x-y+z)$

$\begin{bmatrix} u \\ v \end{bmatrix}$

$= -x + 4y - 3z$.

(c) TO FIND THE KERNEL OF $K \circ L$, JUST SOLVE $(K \circ L)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$:

$$-x + 4y - 3z = 0 \rightsquigarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} -1 & 4 & -3 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{ccc} & \text{FREE} & \text{FREE} \end{array} \\ \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} \end{array}$$

$\therefore$ y, z: FREE

$x = 4y - 3z$

SO THE SOLUTIONS ARE $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4y-3z \\ y \\ z \end{bmatrix}$ WHERE $y, z \in \mathbb{R}$

$\therefore$ $\ker(K \circ L)$ IS JUST $\left\{ \begin{bmatrix} 4y-3z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\}$