1. WITHIN THE CONTEXT OF LINEAR ALGEBRA, WE CALL THE ELEMENTS OF OUR FIELD "SCALARS" (BECAUSE THEY SCALE VECTORS).

2. (a) GIVEN A FIELD F OF SCALARS, A VECTOR SPACE OVER F IS, BASICALLY, A COLLECTION OF OBJECTS ("VECTORS") THAT CAN BE ADDED TO ONE ANOTHER AND MULTIPLIED BY SCALARS.

(GENERALIZING OUR NOTION OF COLUMN VECTORS OVER $\mathbb{R}$, WHICH CAN BE ADDED TO ONE ANOTHER AND SCALED BY ELEMENTS OF $\mathbb{R}$)

FORMALLY, A VECTOR SPACE OVER F IS A SET V OF "VECTORS", ALONG WITH:

- A SUM OPERATION THAT FOR EACH $\vec{v}, \vec{w} \in V$ GIVES US THEIR SUM, $\vec{v}+\vec{w} \in V$

AND
- A SCALAR MULTIPLICATION OPERATION THAT FOR EACH $\alpha \in F$ AND $\vec{v} \in V$ GIVES US THE SCALAR MULTIPLE $\alpha\vec{v} \in V$,

SATISFYING THE FOLLOWING AXIOMS:

$+\begin{cases} \text{(COMMUTATIVE)} & \forall\, \vec{v},\vec{w} \in V,\quad \vec{v}+\vec{w} = \vec{w}+\vec{v} \\ \text{(ASSOCIATIVE)} & \forall\, \vec{v},\vec{w},\vec{u} \in V,\quad \vec{v}+(\vec{w}+\vec{u}) = (\vec{v}+\vec{w})+\vec{u} \\ \text{(IDENTITY)} & \exists\, \vec{0} \in V \text{ SO THAT } \forall\, \vec{v} \in V,\quad \vec{0}+\vec{v} = \vec{v} = \vec{v}+\vec{0} \\ \text{(INVERSES)} & \forall\, \vec{v} \in V\ \exists\, (-\vec{v}) \in V \text{ WITH } \vec{v}+(-\vec{v}) = \vec{0} = (-\vec{v})+\vec{v} \end{cases}$

$\bullet \begin{cases} \text{(UNIT SCALAR)} & \forall\, \vec{v} \in V,\quad 1\vec{v} = \vec{v} \end{cases}$

$+$ AND $\bullet$ $\begin{cases} \text{(DISTRIBUTIVE I)} & \forall\, \alpha \in F \text{ AND } \vec{v},\vec{w} \in V,\quad \alpha(\vec{v}+\vec{w}) = \alpha\vec{v}+\alpha\vec{w} \end{cases}$

COMPATIBILITY WITH $+$ AND $\bullet$ FROM F: $\begin{cases} \forall\, \alpha,\beta \in F \text{ AND } \vec{v} \in V, & \\ (\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v} & \text{(DISTRIBUTIVE II)} \\ \text{AND } (\alpha\cdot\beta)\vec{v} = \alpha(\beta\vec{v}) & \text{(ASSOCIATIVE II)} \end{cases}$

(b) JUST AS WITH COLUMN VECTORS, BEING ABLE TO ADD + SCALE VECTORS ALLOWS US TO FORM LINEAR COMBINATIONS, WHICH ARE OUR FOCUS.

(ADDITION + SCALAR MULTIPLICATION ARE, AS ALWAYS, JUST CONVENIENT BUILDING BLOCKS FOR LINEAR COMBINATIONS)

(c) SOME COMMON VECTOR SPACES ARE:

- F^n, THE VECTOR SPACE OF COLUMN VECTORS OVER F, OF SOME SIZE n.
  (F COULD BE ANY FIELD — $\mathbb{Q}, \mathbb{C}, \mathbb{R}$, ETC.!)

  E.G., $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^3$ (OR $\mathbb{Q}^3$, OR $\mathbb{C}^3$) ; $\begin{bmatrix} 1+i \\ 3i \end{bmatrix} \in \mathbb{C}^2$

  * ANY LIST OF SUMMABLE, SCALABLE QUANTITIES CAN BE VIEWED AS A COLUMN VECTOR!

- $F[x]$, THE VECTOR SPACE OF POLYNOMIALS IN x (WITH COEFFICIENTS IN F), OVER F

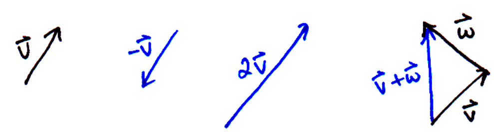
  E.G., $x+x^2+3x^3 \in \mathbb{R}[x]$ (OR $\mathbb{Q}[x]$, OR $\mathbb{C}[x]$)
  $(1+i)+(3-2i)t^2 \in \mathbb{C}[t]$

- $C(\mathbb{R})$, THE VECTOR SPACE OF CONTINUOUS FUNCTIONS $f: \mathbb{R} \to \mathbb{R}$ OR $C(I)$, THE VECTOR SPACE OF CONTINUOUS FUNCTIONS $f: I \to \mathbb{R}$

  E.G., $f(x) = \sin x$ IS IN $C(\mathbb{R})$
  $g(x) = \frac{1}{x}$ IS IN $C([1,2])$

  * ANY CONTINUOUS FUNCTION CAN BE CONSIDERED AS SUCH A VECTOR!

- GEOMETRIC VECTORS OF SOME DIMENSION, OVER $\mathbb{R}$, WHICH HAVE MAGNITUDE (LENGTH) AND DIRECTION, BUT NO FIXED POSITION — THEY CAN BE SCALED AND ADDED (HEAD-TO-TAIL), AND THEY SATISFY ALL VECTOR SPACE AXIOMS



(d) A SUBSPACE OF A VECTOR SPACE V IS A SUBCOLLECTION $W \subset V$ THAT IS A VECTOR SPACE IN ITS OWN RIGHT — I.E., IF WE FORM LINEAR COMBINATIONS OF VECTORS IN W, WE GET VECTORS OF W.

TO CHECK THAT W IS A SUBSPACE OF V, WE ONLY NEED TO VERIFY THAT:

⓪ $\vec{0} \in W$

① $\forall\, \alpha \in F$ AND $\vec{w}_1 \in W$, $\alpha \vec{w}_1 \in W$

② $\forall\, \vec{w}_1, \vec{w}_2 \in W$, $\vec{w}_1 + \vec{w}_2 \in W$

(IF THESE THREE CONDITIONS HOLD, THEN BREAKING DOWN ANY LINEAR COMBINATION OF VECTORS IN W INTO SUMS OF SCALAR MULTIPLES LETS US CONCLUDE THAT THE L.C. MUST END UP IN W — IN THIS CASE, THE VECTOR SPACE AXIOMS FOR V SHOW US THAT W SATISFIES ALL AXIOMS OF A VECTOR SPACE).

3. (a) WE COULD USE COLUMN VECTORS TO REPRESENT THE QUANTITIES OF CARBON, HYDROGEN AND OXYGEN IN EACH MOLECULE, OBTAINING:

$$\begin{array}{c} \\ C \rightarrow \\ H \rightarrow \\ O \rightarrow \end{array} \begin{array}{c} CH_4 \\ \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \end{array} + 2 \begin{array}{c} O_2 \\ \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \end{array} = \begin{array}{c} CO_2 \\ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \end{array} + 2 \begin{array}{c} H_2O \\ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \end{array}$$

(SETTING UP SUCH A LINEAR SYSTEM IN SUCH A WAY, WE CAN SYSTEMATICALLY BALANCE ANY CHEMICAL REACTION!)

(b) WE COULD AGAIN USE COLUMN VECTORS TO REPRESENT THIS INFORMATION, WITH THE ENTRIES GIVING THE PROPORTIONS OF HIGH-, MEDIUM-, AND LOW-RISK COMPONENTS, E.G.:

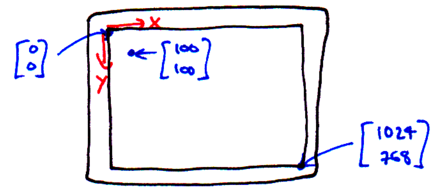
$$\begin{array}{r} \\ \text{HIGH} \rightarrow \\ \text{MEDIUM} \rightarrow \\ \text{LOW} \rightarrow \end{array} \begin{array}{c} \text{A VERY HIGH-RISK FUND} \\ \begin{bmatrix} 100\% \\ 0\% \\ 0\% \end{bmatrix} \end{array} ; \begin{array}{c} \text{A SOMEWHAT RISK-BALANCED FUND} \\ \begin{bmatrix} 50\% \\ 30\% \\ 20\% \end{bmatrix} \end{array} ; \begin{array}{c} \text{A LOW-RISK FUND} \\ \begin{bmatrix} 0\% \\ 20\% \\ 80\% \end{bmatrix} \end{array}$$

(WE COULD FORM LINEAR COMBINATIONS OF THESE FUNDS TO ACHIVE SOME DESIRED TARGET RISK-PROFILE)

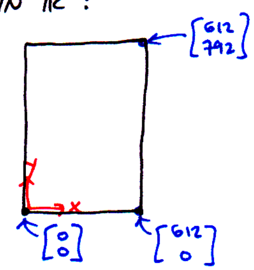
(c) WAVEFORMS CAN BE VIEWED SIMPLY AS VECTORS IN $C(\mathbb{R})$ — (E.G., $f(x) = \sin x$, $g(x) = \cos 3x$, $h(x) = 4\cos 2x - \sin 5x$) COMBINING AND AMPLIFYING WAVEFORMS SIMPLY AMOUNTS TO FORMING LINEAR COMBINATIONS OF THESE VECTORS!

E.G., $3(f + g + h)$, WITH f, g, h AS ABOVE

(d) A LOCATION ON A COMPUTER DISPLAY OR PIECE OF PAPER CAN (AFTER CHOOSING AXES, UNITS, AND AN ORIGIN) BE VIEWED AND MANIPULATED SIMPLY AS VECTORS IN $\mathbb{R}^2$:

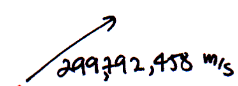


(A 1024×768 PIXEL DISPLAY)

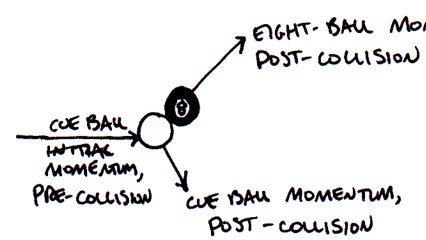


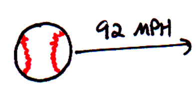
(AN $8\frac{1}{2} \times 11''$ PIECE OF PAPER, MEASURED IN POINTS ($\frac{1}{72}''$))

(e) VELOCITY AND MOMENTUM (HAVING BOTH MAGNITUDE + DIRECTION) ARE NATURALLY REPRESENTED BY GEOMETRIC VECTORS, AFTER CHOOSING UNITS FOR TIME + SPACE (AND MASS, IN THE CASE OF MOMENTUM):



A PHOTON





A BASEBALL

4. (a) $2i\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1+i\\-3\end{bmatrix} + i\begin{bmatrix}0\\i\end{bmatrix} = \begin{bmatrix}2i\\2i\end{bmatrix} - \begin{bmatrix}1+i\\-3\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}$

$= \begin{bmatrix}2i-(1+i)+0\\2i-(-3)+(-1)\end{bmatrix} = \begin{bmatrix}2i-1-i\\2i+3-1\end{bmatrix} = \underline{\underline{\begin{bmatrix}-1+i\\2+2i\end{bmatrix}}}$

(b) $0\begin{bmatrix}5i\\6+i\end{bmatrix} + 5\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}-2\\3i\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}-2\\3i\end{bmatrix} = \underline{\underline{\begin{bmatrix}-2\\3i\end{bmatrix}}}$

(c) $(4-i)\begin{bmatrix}2+i\\1\end{bmatrix} - 3\begin{bmatrix}-1+i\\5-2i\end{bmatrix} = \begin{bmatrix}9+2i\\4-i\end{bmatrix} + \begin{bmatrix}3-3i\\-15+6i\end{bmatrix} = \underline{\underline{\begin{bmatrix}12-i\\-11+5i\end{bmatrix}}}$

5. (a) $-(1+x+x^2)+2(4+x) = -1-x-x^2+8+2x = \underline{\underline{7+x-x^2}}$

(b) $1-(1+x)+(x+x^2) = 1-1-x+x+x^2 = \underline{\underline{x^2}}$

(c) $2(1-x^2)-4(1-x+x^2) = 2-2x^2-4+4x-4x^2 = \underline{\underline{-2+4x-6x^2}}$

6. (a) $-(1-t+t^2)+2(1-t) = -1+t-t^2+2-2t = 1-t-t^2 = \underline{\underline{1+2t+2t^2}}$

(b) $2(1+t)-(1-2t+2t^2) = 2+2t-1+2t-2t^2$

$= 1+4t-2t^2 = \underline{\underline{1+t+t^2}}$

(c) $(t^2+t^4)+1-2(1+t^2) = t^2+t^4+1-2-2t^2$

$= -1-t^2+t^4 = \underline{\underline{2+2t^2+t^4}}$

(SUBSPACE CHECKLIST: 0, 1, 2 FROM PROBLEM 2(d))

7. (a) IS $W = \left\{\begin{bmatrix}x\\y\\0\end{bmatrix} : x,y\in\mathbb{R}\right\}$ A SUBSPACE OF $\mathbb{R}^3$?

0 TAKING $x=0$, $y=0 \in \mathbb{R}$, WE HAVE $\vec{0} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}x\\y\\0\end{bmatrix} \in W$ ✓

1 SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $\vec{w}_1 \in W$;

THEN $\vec{w}_1 = \begin{bmatrix}x_1\\y_1\\0\end{bmatrix}$ FOR SOME $x_1, y_1 \in \mathbb{R}$.

NOW, $\alpha\vec{w}_1 = \begin{bmatrix}\alpha x_1\\\alpha y_1\\0\end{bmatrix}$, SO TAKING $x = \alpha x_1 \in \mathbb{R}$ AND $y = \alpha y_1 \in \mathbb{R}$,

WE SEE THAT $\alpha\vec{w}_1 = \begin{bmatrix}x\\y\\0\end{bmatrix} \in W$ ✓

2 SUPPOSE THAT $\vec{w}_1, \vec{w}_2 \in W$;

THEN $\vec{w}_1 = \begin{bmatrix}x_1\\y_1\\0\end{bmatrix}$ AND $\vec{w}_2 = \begin{bmatrix}x_2\\y_2\\0\end{bmatrix}$ FOR SOME $x_1, y_1, x_2, y_2 \in \mathbb{R}$,

SO $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix}x_1+x_2\\y_1+y_2\\0\end{bmatrix}$.

THUS, TAKING $x = x_1 + x_2 \in \mathbb{R}$ AND $y = y_1 + y_2 \in \mathbb{R}$, WE SEE THAT $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix}x\\y\\0\end{bmatrix} \in W$ ✓

$\therefore$ YES — THIS IS A SUBSPACE OF $\mathbb{R}^3$!

(b) IS $W = \left\{ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x,y \in \mathbb{R} \right\}$ A SUBSPACE OF $\mathbb{R}^3$?

⓪ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ FOR ANY $x,y \in \mathbb{R}$, SO $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$ ✗

NO — THIS IS NOT A SUBSPACE OF $\mathbb{R}^3$!

(c) IS $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y,z \in \mathbb{R} \text{ AND } z = x+y \right\}$ A SUBSPACE OF $\mathbb{R}^3$?

⓪ TAKING $x=y=z=0 \in \mathbb{R}$, WE HAVE $z = x+y$, SO

$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ ✓

① SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $\vec{w}_1 \in W$; THEN $\vec{w}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ FOR SOME

$x_1, y_1, z_1 \in \mathbb{R}$ WITH $z_1 = x_1 + y_1$. NOW, $\alpha \vec{w}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{bmatrix}$, SO

TAKING $x = \alpha x_1$, $y = \alpha y_1$, $z = \alpha z_1 \in \mathbb{R}$,

WE HAVE $z = \alpha z_1 = \alpha(x_1 + y_1) = \alpha x_1 + \alpha y_1 = x + y$,

SO $\alpha \vec{w}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ ✓

② SUPPOSE THAT $\vec{w}_1, \vec{w}_2 \in W$; THEN $\vec{w}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ AND $\vec{w}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

FOR SOME $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$ WITH $z_1 = x_1 + y_1$ AND $z_2 = x_2 + y_2$.

NOW, $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$, SO TAKING $x = x_1 + x_2$, $y = y_1 + y_2$, $z = z_1 + z_2$,

WE HAVE $x,y,z \in \mathbb{R}$ AND

$$z = z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) = x + y,$$

SO $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ ✓

YES — THIS IS A SUBSPACE OF $\mathbb{R}^3$!

(d) IS $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y,z \in \mathbb{R} \text{ AND } [x=0 \text{ OR } y=0] \right\}$ A SUBSPACE OF $\mathbb{R}^3$?

↳ NOTE: IN MATHEMATICS, "P OR Q" ALWAYS MEANS EITHER P, OR Q, OR BOTH.

⓪ TAKING $x=y=z=0 \in \mathbb{R}$, IT IS TRUE THAT $[x=0 \text{ OR } y=0]$,

SO $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ ✓

① SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $\vec{w}_1 \in W$;

THEN $\vec{w}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ FOR SOME $x_1, y_1, z_1 \in \mathbb{R}$ WITH $[x_1 = 0 \text{ OR } y_1 = 0]$.

NOW, $\alpha \vec{w}_1 = \begin{bmatrix} \alpha x_1 \\ \alpha y_1 \\ \alpha z_1 \end{bmatrix}$, SO TAKING $x = \alpha x_1$, $y = \alpha y_1$, $z = \alpha z_1 \in \mathbb{R}$,

WE HAVE $\alpha \vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, AND SINCE $[x_1 = 0 \text{ OR } y_1 = 0]$,

WE KNOW $[\alpha x_1 = 0 \text{ OR } \alpha y_1 = 0]$,

I.E., $[x=0 \text{ OR } y=0]$,

SO $\alpha \vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ ✓

② SUPPOSE THAT $\vec{w}_1, \vec{w}_2 \in W$. THEN $\vec{w}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ AND $\vec{w}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

FOR SOME $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$ WITH $[x_1 = 0 \text{ OR } y_1 = 0]$

AND $[x_2 = 0 \text{ OR } y_2 = 0]$.

NOW, $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$, SO TO HAVE $\vec{w}_1 + \vec{w}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

WE MUST TAKE $x = x_1 + x_2$, $y = y_1 + y_2$, $z = z_1 + z_2 \in \mathbb{R}$.

FOR $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ TO BE IN W, WE MUST HAVE $[x=0 \text{ OR } y=0]$.

— BUT $x = x_1 + x_2$ AND $y = y_1 + y_2$, AND ALL THAT WE KNOW IS THAT $[x_1 = 0 \text{ OR } y_1 = 0]$ AND $[x_2 = 0 \text{ OR } y_2 = 0]$.

* WE CAN'T CONCLUDE WITH CERTAINTY THAT $x=0$ OR $y=0$! HOW DO WE MAKE THIS EXPLICIT? WE FIND SOME VALUES $x_1, y_1, x_2, y_2 \in \mathbb{R}$ WITH $[x_1=0 \text{ OR } y_1=0]$ AND $[x_2=0 \text{ OR } y_2=0]$

BUT NOT $[x_1+x_2=0 \text{ OR } y_1+y_2=0]$.

E.G., TAKE $x_1=y_2=0$ AND $x_2=y_1=1$.

IT IS FALSE THAT $\forall \vec{\omega}_1, \vec{\omega}_2 \in W$, $\vec{\omega}_1+\vec{\omega}_2 \in W$ — FOR EXAMPLE,

TAKE $\vec{\omega}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ AND $\vec{\omega}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. THESE ARE IN W BECAUSE $[0=0 \text{ OR } 1=0]$ + $[1=0 \text{ OR } 0=0]$, BUT

$\vec{\omega}_1+\vec{\omega}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin W$ BECAUSE IT IS NOT TRUE THAT $[1=0 \text{ OR } 1=0]$.

NO — THIS IS NOT A SUBSPACE OF $\mathbb{R}^3$! ✗

(e) IS $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x,y,z \in \mathbb{R} \text{ AND } x \neq 0 \right\}$ A SUBSPACE OF $\mathbb{R}^3$?

(0) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$, BECAUSE IT IS FALSE THAT $0 \neq 0$ ✗

NO — THIS IS NOT A SUBSPACE OF $\mathbb{R}^3$.

8. (a) IS $W = \{1 + Ax : A \in \mathbb{R}\}$ A SUBSPACE OF $\mathbb{R}[x]$?

(0) $\vec{0} = 0 \notin W$, BECAUSE FOR NO $A \in \mathbb{R}$ DOES $0 = 1 + Ax$. ✗

NOTE THAT IN W, A IS WHAT VARIES

— x IS JUST THE VARIABLE USED IN THE POLYNOMIAL.

NO — THIS IS NOT A SUBSPACE OF $\mathbb{R}[x]$.

(b) IS $W = \{C + Cx : C \in \mathbb{R}\}$ A SUBSPACE OF $\mathbb{R}[x]$?

(0) TAKING $C = 0 \in \mathbb{R}$, $\vec{0} = 0 + 0x = C + Cx \in W$ ✓

(1) SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $\vec{\omega}_1 \in W$;
THEN $\vec{\omega}_1 = C_1 + C_1 x$ FOR SOME $C_1 \in \mathbb{R}$.
NOW, $\alpha\vec{\omega}_1 = \alpha(C_1 + C_1 x) = (\alpha C_1) + (\alpha C_1)x$,
SO TAKING $C = \alpha C_1$, WE HAVE $\alpha\vec{\omega}_1 = C + Cx \in W$ ✓

(2) SUPPOSE THAT $\vec{\omega}_1, \vec{\omega}_2 \in W$;
THEN $\vec{\omega}_1 = C_1 + C_1 x$ AND $\vec{\omega}_2 = C_2 + C_2 x$ FOR SOME $C_1, C_2 \in \mathbb{R}$.
NOW, $\vec{\omega}_1 + \vec{\omega}_2 = (C_1 + C_1 x) + (C_2 + C_2 x) = (C_1 + C_2) + (C_1 + C_2)x$,
SO TAKING $C = C_1 + C_2$, WE HAVE $\vec{\omega}_1 + \vec{\omega}_2 = C + Cx \in W$ ✓

YES — THIS IS A SUBSPACE OF $\mathbb{R}[x]$!

(c) IS $W = \{A + Bx + Cx^2 : A, B, C \in \mathbb{R}\}$ A SUBSPACE OF $\mathbb{R}[x]$?

⓪ TAKING $A = B = C = 0 \in \mathbb{R}$, $\vec{0} = 0 + 0x + 0x^2 = A + Bx + Cx^2 \in W$ ✓

① SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $\vec{\omega}_1 \in W$; THEN $\vec{\omega}_1 = A_1 + B_1 x + C_1 x^2$ FOR SOME $A_1, B_1, C_1 \in \mathbb{R}$.

NOW, $\alpha\vec{\omega}_1 = \alpha(A_1 + B_1 x + C_1 x^2) = \alpha A_1 + \alpha B_1 x + \alpha C_1 x^2$,
SO TAKING $A = \alpha A_1$, $B = \alpha B_1$, $C = \alpha C_1 \in \mathbb{R}$,
WE HAVE $\alpha\vec{\omega}_1 = A + Bx + Cx^2 \in W$ ✓

② SUPPOSE THAT $\vec{\omega}_1, \vec{\omega}_2 \in W$;
THEN $\vec{\omega}_1 = A_1 + B_1 x + C_1 x^2$ AND $\vec{\omega}_2 = A_2 + B_2 x + C_2 x^2$,
FOR SOME $A_1, B_1, C_1, A_2, B_2, C_2 \in \mathbb{R}$.

NOW, $\vec{\omega}_1 + \vec{\omega}_2 = (A_1 + B_1 x + C_1 x^2) + (A_2 + B_2 x + C_2 x^2)$
$= (A_1 + A_2) + (B_1 + B_2)x + (C_1 + C_2)x^2$,
SO TAKING $A = A_1 + A_2$, $B = B_1 + B_2$, $C = C_1 + C_2 \in \mathbb{R}$, WE HAVE
$\vec{\omega}_1 + \vec{\omega}_2 = A + Bx + Cx^2 \in W$ ✓

YES — THIS IS A SUBSPACE OF $\mathbb{R}[x]$!

9. (a) IS $W = \{f \in C(\mathbb{R}) : f(0) = 0\}$ A SUBSPACE OF $C(\mathbb{R})$?

⓪ $\vec{0} \in C(\mathbb{R})$ IS THE "ZERO FUNCTION" $f : x \mapsto 0$.
WELL, THIS FUNCTION CERTAINLY SATISFIES $f(0) = 0$, SO $\vec{0} \in W$ ✓

① SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $f_1 \in W$; THEN $f_1 \in C(\mathbb{R})$ AND $f_1(0) = 0$.
WELL, $f = \alpha f_1 \in C(\mathbb{R})$ TOO, AND $f(0) = (\alpha f_1)(0) = \alpha f_1(0) = \alpha \cdot 0 = 0$,
SO $\alpha f_1 = f \in W$. ✓

② SUPPOSE THAT $f_1, f_2 \in W$; THEN $f_1, f_2 \in C(\mathbb{R})$ HAVE
$f_1(0) = 0$ AND $f_2(0) = 0$.
WELL, $f = f_1 + f_2 \in C(\mathbb{R})$ TOO, AND
$f(0) = (f_1 + f_2)(0) = f_1(0) + f_2(0) = 0 + 0 = 0$,
SO $f_1 + f_2 = f \in W$ ✓

YES — THIS IS A SUBSPACE OF $C(\mathbb{R})$!

(b) IS $W = \{f \in C(\mathbb{R}) : f(0) = 2\}$ A SUBSPACE OF $C(\mathbb{R})$?

⓪ AGAIN, $\vec{0} \in C(\mathbb{R})$ IS THE "ZERO FUNCTION" $f : x \mapsto 0$.
BUT THEN $f(0) = 0 \neq 2$, SO $\vec{0} \notin W$ ✗

NO — THIS IS NOT A SUBSPACE OF $C(\mathbb{R})$.

(c) IS $W = \{f \in C(\mathbb{R}) : f(0) \neq 1\}$ A SUBSPACE OF $C(\mathbb{R})$?

⓪ $\vec{0} \in C(\mathbb{R})$ IS $f : x \mapsto 0$,
SO BECAUSE $f(0) = 0 \neq 1$, $\vec{0} = f \in W$ ✓

① SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $f_1 \in W$; THEN $f_1 \in C(\mathbb{R})$ HAS $f_1(0) \neq 1$.
WELL, $\alpha f_1 \in C(\mathbb{R})$, AND $(\alpha f_1)(0) = \alpha f_1(0) \overset{?}{\neq} 1$

* WE CAN'T CONCLUDE WITH CERTAINTY THAT $\alpha f_1(0) \neq 1$;
TO SHOW THIS EXPLICITLY, WE MUST FIND SOME $\alpha \in \mathbb{R}$
AND SOME $f_1 \in C(\mathbb{R})$ WITH $f_1(0) \neq 1$ BUT $\alpha f_1(0) = 1$...

TAKE, E.G., $f_1 : x \mapsto 2$ AND $\alpha = \frac{1}{2}$.
THEN $f_1 \in W$ (BECAUSE $f_1 \in C(\mathbb{R})$ AND $f_1(0) = 2 \neq 1$)
BUT $f = \alpha f_1 \notin W$,
BECAUSE $f(0) = (\alpha f_1)(0) = \alpha f_1(0) = \frac{1}{2} \cdot 2 = 1$. ✗

NO — THIS IS NOT A SUBSPACE OF $C(\mathbb{R})$.

(d) IS $W = \{f \in C(\mathbb{R}) : f(0) = 2f(1) \text{ AND } f(3) = 0\}$
A SUBSPACE OF $C(\mathbb{R})$?

(0) AGAIN, $\vec{0} \in C(\mathbb{R})$ IS THE FUNCTION $f : x \mapsto 0$,
AND FOR THIS FUNCTION, $f(0) = 0 = 2 \cdot 0 = 2f(1)$ AND $f(3) = 0$,
SO $\vec{0} = f \in W$ ✓

(1) SUPPOSE THAT $\alpha \in \mathbb{R}$ AND $f_1 \in W$;
THEN $f_1 \in C(\mathbb{R})$ HAS $f_1(0) = 2f_1(1)$ AND $f_1(3) = 0$.
WELL, $f = \alpha f_1 \in C(\mathbb{R})$ THEN HAS
$f(0) = (\alpha f_1)(0) = \alpha f_1(0) = \alpha(2f_1(1)) = 2\alpha f_1(1) = 2(\alpha f_1)(1) = 2f(1)$
AND $f(3) = (\alpha f_1)(3) = \alpha f_1(3) = \alpha \cdot 0 = 0$,
SO $\alpha f_1 = f \in W$ ✓

(2) SUPPOSE THAT $f_1, f_2 \in W$;
THEN $f_1, f_2 \in C(\mathbb{R})$ HAVE $f_1(0) = 2f_1(1)$, $f_1(3) = 0$,
$f_2(0) = 2f_2(1)$, AND $f_2(3) = 0$.
WELL, $f = f_1 + f_2 \in C(\mathbb{R})$ THEN HAS

$$f(0) = (f_1 + f_2)(0) = f_1(0) + f_2(0) = 2f_1(1) + 2f_2(1)$$
$$= 2(f_1(1) + f_2(1)) = 2(f_1 + f_2)(1) = 2f(1)$$

AND $f(3) = (f_1 + f_2)(3) = f_1(3) + f_2(3) = 0 + 0 = 0$,

SO $f_1 + f_2 = f \in W$ ✓

YES — THIS IS A SUBSPACE OF $C(\mathbb{R})$!