1. A PROOF OF A LOGICAL STATEMENT IS A LOGICALLY INESCAPABLE ARGUMENT SHOWING THAT THE STATEMENT MUST BE TRUE; IT CONSISTS OF A SEQUENCE OF STRUCTURED LOGICAL ASSERTIONS, WITH THE TRUTH OF EACH ASSERTION JUSTIFIED BY SOME DEFINITION, HYPOTHESIS, RULE, THEOREM, OR PRIOR ASSERTION IN THE PROOF.

2. PROOFS ARE POSSIBLE IN MATHEMATICS BECAUSE EVERY MATHEMATICAL OBJECT HAS A DEFINITION AND A SET OF RULES (AXIOMS) ACCORDING TO WHICH IT BEHAVES.

3. PROOFS ALLOW US TO BE CERTAIN THAT PARTICULAR LOGICAL RELATIONSHIPS HOLD IN MATHEMATICS, SO THAT WE CAN SECURELY BUILD UPON WHAT WE'VE ALREADY ESTABLISHED; BUILDING FROM BASIC PRINCIPLES TO DEEP AND SIGNIFICANT RESULTS IS THE POINT OF MATHEMATICS, AND WITHOUT PROOFS THIS WOULD NOT BE POSSIBLE.

4. THE STATEMENT $P \Rightarrow Q$: A.K.A. "P IMPLIES Q" OR "IF P, THEN Q"

(a) P IS THE HYPOTHESIS OF THIS STATEMENT; Q IS THE CONCLUSION.

(b) THE MOST DIRECT WAY TO PROVE AN IMPLICATION (IF-THEN STATEMENT) IS TO START BY SUPPOSING THE HYPOTHESIS AND USE THIS TO REASON ONE STEP AT A TIME TO THE CONCLUSION.

E.G., CLAIM: IF $\underbrace{\sin x = 0}_{\text{HYPOTHESIS}}$, THEN $\underbrace{\sin 2x = 0}_{\text{CONCLUSION}}$.

PROOF: SUPPOSE THAT $\sin x = 0$.

THEN $\sin 2x = 2 \sin x \cos x = 2 \cdot 0 \cdot \cos x = 0$ ∎

(c) A COUNTEREXAMPLE TO AN IF-THEN STATEMENT IS AN EXAMPLE IN WHICH: ① THE HYPOTHESIS IS TRUE, BUT ② THE CONCLUSION IS FALSE.

THE EXISTENCE OF A COUNTEREXAMPLE TO AN IF-THEN STATEMENT DEMONSTRATES THAT THE IF-THEN STATEMENT IS FALSE.

(d) THE STATEMENT "$P \Rightarrow Q$" IS LOGICALLY EQUIVALENT TO ITS CONTRAPOSITIVE, THE STATEMENT "NOT Q $\Rightarrow$ NOT P."

5. THE STATEMENT $\forall x, Q$: A.K.A. "FOR ALL x, Q"

(a) THE MOST DIRECT WAY TO PROVE SUCH A FOR-ALL STATEMENT IS TO LET AN UNKNOWN x BE GIVEN AND THEN REASON ONE STEP AT A TIME TO THE TRUTH OF THE STATEMENT Q.

E.G., CLAIM: $\forall x$, $0 \cdot x = 0$

PROOF: LET x BE GIVEN.

THEN $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$

SUBTRACTING $0 \cdot x$ FROM BOTH SIDES,

$0 = 0 \cdot x$. ∎

(b) TO DEMONSTRATE THAT THE STATEMENT "$\forall x, Q$" IS FALSE, ALL THAT IS NEEDED IS TO FIND JUST ONE VALUE OF x FOR WHICH THE STATEMENT Q IS FALSE.

6. THE STATEMENT $\exists x$ SUCH THAT Q: A.K.A. "THERE EXISTS x SUCH THAT Q"

OR "FOR WHICH", "SO THAT", ETC.

... THESE WORDS ARE JUST FILLERS!

(a) THE MOST DIRECT WAY TO PROVE SUCH A THERE-EXISTS STATEMENT IS TO (ANY WAY YOU CAN) FIND SUCH AN x, THEN TAKE THAT x IN YOUR PROOF AND, FINALLY, REASON ONE STEP AT A TIME TO THE TRUTH OF THE STATEMENT Q.

E.G., CLAIM: $\exists x$ SUCH THAT $x^2 - x - 1 = 0$.

SCRATCH WORK:

$x^2 - x - 1 = 0$

$\Rightarrow x = \dfrac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

PROOF: TAKE $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

THEN $x^2 - x - 1 = \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^2 - \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) - 1$

$= \cancel{\frac{1}{4}} + \cancel{\frac{1}{2}\sqrt{5}} + \cancel{\frac{5}{4}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}\sqrt{5}} - \cancel{1}$

$= 0$. ∎

(b) TO DEMONSTRATE THAT THE STATEMENT "$\exists x$ SUCH THAT Q" IS FALSE, SHOW THAT FOR ALL x, (Q IS FALSE).

7. (a) A SET IS A COLLECTION OF OBJECTS, CALLED ITS ELEMENTS; TWO SETS ARE CONSIDERED TO BE EQUAL WHEN THEY CONTAIN EXACTLY THE SAME ELEMENTS.

(b) THE NOTATION $a \in A$ MEANS THAT a IS AN ELEMENT OF A; FROM CONTEXT, THIS STATEMENT TELLS US THAT A IS A SET AND THAT a IS AN ~~be~~ ELEMENT OF THAT SET.

(c) A SET CAN BE PRESENTED EXPLICITLY BY LISTING ITS ELEMENTS (USING BRACES TO DELIMIT THE SET AND COMMAS OR SEMICOLONS BETWEEN THE ELEMENTS), FOR EXAMPLE:

$$\{0, 1, 5, 7, 10\} \quad \text{or} \quad \{\text{RED, GREEN, BLUE}\}.$$

WE SOMETIMES USE ELLIPSES ("...") TO INDICATE A CONTINUING LIST (BUT USE THEM CAREFULLY!), E.G.,

↳ BECAUSE THERE IS A DANGER OF MISINTERPRETATION WHEN ELLIPSES ARE EMPLOYED!

$$\{1, 2, 3, \ldots, 100\} \quad \text{or} \quad \{\ldots, -3, -2, -1, 0\}.$$

A SET CAN ALSO BE PRESENTED IMPLICITLY, BY GIVING THE FORM OF THE SET'S ELEMENTS (ON THE LEFT) AND THE CREDENTIALS REQUIRED FOR INCLUSION IN THE SET (ON THE RIGHT). AS BEFORE, BRACES DELIMIT THE SET, AND WE USE EITHER A COLON OR A VERTICAL BAR TO SEPARATE THE TWO SIDES.

FOR EXAMPLE:

$$\{x \in \mathbb{R} : \sin x = 0\} = \{\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$$

$$\text{or} \quad \{2k+1 \mid k \in \mathbb{Z}\} = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$$

(d) $\emptyset$ IS THE EMPTY SET $\{\}$, THE SET WITH NO ELEMENTS AT ALL.

$\mathbb{N}$ IS THE SET OF NATURAL NUMBERS, $\{1, 2, 3, 4, 5, \ldots\}$.

$\mathbb{Z}$ IS THE SET OF INTEGERS, $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

$\mathbb{Q}$ IS THE SET OF RATIONALS, $\{p/q : p, q \in \mathbb{Z} \text{ AND } q \neq 0\}$.

$\mathbb{R}$ IS THE SET OF REAL NUMBERS.

$\mathbb{C}$ IS THE SET OF COMPLEX NUMBERS, $\{a + bi : a, b \in \mathbb{R}\}$ (WHERE $i^2 = -1$).

(e) IF S IS A SET, $|S|$ REPRESENTS THE CARDINALITY OF S; WHEN S HAS A FINITE NUMBER OF ELEMENTS, THIS CAN BE THOUGHT OF AS THAT NUMBER (THE "SIZE") OF THE SET; E.G., $|\emptyset| = 0$, $|\{a, b, a\}| = 2$, ETC.

(f) $A \subset B$ MEANS THAT A AND B ARE SETS, AND THAT EVERY ELEMENT OF A IS AN ELEMENT OF B (IN SYMBOLS, $x \in A \Rightarrow x \in B$); INFORMALLY, THE SET A "FITS INSIDE" THE SET B.

IN THIS CASE, WE SAY THAT A IS A SUBSET OF B.

8. THEY'RE ALL DIFFERENT WAYS OF EXPRESSING THE SAME SET, THE SET OF EVEN INTEGERS. NO MATTER WHETHER A SET IS EXPRESSED IMPLICITLY (AS THE FIRST TWO) OR EXPLICITLY (AS THE LAST TWO), NOR WHAT VARIABLES OR OPERATIONS ARE USED, WHAT DETERMINES A SET IS SIMPLY WHAT ITS ELEMENTS ARE!

9. THE ALGEBRAIC CONCEPT OF FIELD SIMPLY GENERALIZES THE IDEA OF A SET OF NUMBERS FOR WHICH ARITHMETIC $(+, -, \times, \div)$ WORKS AS USUAL.

FORMALLY, A FIELD IS A SET F EQUIPPED WITH TWO BINARY OPERATIONS OF ADDITION $(+)$ AND MULTIPLICATION $(\cdot)$ ON F SATISFYING THE FOLLOWING AXIOMS (THE FIELD AXIOMS):

AXIOMS FOR $+$

- $\forall x, y \in F, \quad x+y=y+x$ (COMMUTATIVITY OF +)
- $\forall x, y, z \in F, \quad x+(y+z)=(x+y)+z$ (ASSOCIATIVITY OF +)
- $\exists\, 0 \in F$ SUCH THAT $\forall x \in F,\ x+0=x=0+x$ (IDENTITY FOR +)
- $\forall x \in F\ \exists\, (-x) \in F$ WITH $x+(-x)=0=(-x)+x$ (INVERSES FOR +)

AXIOMS FOR $\cdot$

- $\forall x, y \in F, \quad x \cdot y = y \cdot x$ (COMMUTATIVITY OF ·)
- $\forall x, y, z \in F, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (ASSOCIATIVITY OF ·)
- $\exists\, 1 \in F,\ 1 \neq 0$, SO THAT $\forall x \in F,\ 1 \cdot x = x = x \cdot 1$ (IDENTITY FOR ·)
- $\forall x \in F$ WITH $x \neq 0$, $\exists\, x^{-1} \in F$ WITH $x \cdot (x^{-1}) = 1 = (x^{-1}) \cdot x$ (INVERSES FOR ·)

AND THE DISTRIBUTIVE LAW $\{\forall x, y, z \in F, \quad x \cdot (y+z) = x \cdot y + x \cdot z$

* IN THIS COURSE, WE WILL NOT BE FOCUSING ON THIS AXIOMATIC TREATMENT OF FIELDS — BUT IT IS IMPORTANT TO NOTE THAT EVERY ARITHMETIC PROPERTY WE EXPECT OF $\mathbb{R}$ (FOR INSTANCE) IS A CONSEQUENCE OF THESE NINE AXIOMS!

A FEW COMMON FIELDS ARE:

- $\mathbb{Q}$, THE FIELD OF RATIONALS ($\frac{p}{q}$, WHERE p AND q ARE INTEGERS WITH $q \neq 0$) → ARITHMETIC AS USUAL
- $\mathbb{R}$, THE FIELD OF REALS
- $\mathbb{C}$, THE FIELD OF COMPLEX NUMBERS ($a+bi$, WHERE $a, b \in \mathbb{R}$ AND $i^2 = -1$)

ARITHMETIC IS AS IF i WERE A VARIABLE, BUT EXPLOITING THE FACT THAT $i^2=-1$ FOR $\cdot + \div$, E.G.:

$$(2+i)(3-4i) = 6-5i-4i^2 = 6-5i+4 = 10-5i$$

$$\frac{2+i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{2+11i}{3^2-(4i)^2} = \frac{2+11i}{9+16} = \frac{2}{25} + \frac{11}{25}i$$

AND EVEN

- $\mathbb{Z}_p$, THE FIELD OF "INTEGERS MOD p", WHERE p IS A PRIME.

(WE WON'T BE DISCUSSING THIS FIELD IN OUR COURSE, BUT IF YOU'RE CURIOUS, YOU CAN CHECK THAT ALL FIELD AXIOMS ARE ACTUALLY SATISFIED!!!)

10. (a) $-1 + \frac{1}{8} = -\frac{8}{8} + \frac{1}{8} = \boxed{-\frac{7}{8}}$

$\left(-\frac{3}{7}\right)^{-1} = \boxed{-\frac{7}{3}}$

$$\frac{5 + \frac{2}{3}}{\frac{3}{4} - \frac{1}{5}} = \frac{\frac{15}{3} + \frac{2}{3}}{\frac{15}{20} - \frac{4}{20}} = \frac{17/3}{11/20} = \frac{17}{3} \cdot \frac{20}{11} = \boxed{\frac{340}{33}}$$

(b) $(2+i) - (3-i) = 2 + i - 3 + i = \boxed{-1 + 2i}$

$$\frac{1+i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3+3i-4i-4i^2}{3^2-(4i)^2} = \frac{3-i+4}{9+16} = \frac{7-i}{25} = \boxed{\frac{7}{25} - \frac{1}{25}i}$$

$$(1+i)^4 = ((1+i)^2)^2 = (\cancel{1} + 2i + \cancel{i^2})^2 = (2i)^2 = 4i^2 = \boxed{-4}$$

(c) (IN $\mathbb{Z}_2$, WE CONSIDER ALL EVEN INTEGERS TO BE 0.)

$0 \times 1 = \boxed{0}$

$1 \times (1 + 0 + 1) = 1 \times (2) = 1 \cdot 0 = \boxed{0}$

$\left(\frac{1}{1+1+1}\right)^2 = \left(\frac{1}{1+2}\right)^2 = \left(\frac{1}{1}\right)^2 = 1^2 = \boxed{1}$

(d) (IN $\mathbb{Z}_5$, WE CONSIDER ALL MULTIPLES OF 5 TO BE 0.)

$-3 = 2-5 = 2-0 = \boxed{2}$

$\frac{1}{2} = \boxed{3}$ (SINCE $2 \cdot 3 = 6 = 5+1 = 1$)

$1^4 = \boxed{1}$

$2^4 = 16 = 15+1 = \boxed{1}$

$3^4 = 81 = 80+1 = \boxed{1}$

$4^4 = 256 = 255+1 = \boxed{1}$