

1. A LINEAR SYSTEM IS A COLLECTION OF EQUATIONS, IN SOME NUMBER OF VARIABLES, EACH CONTAINING ONLY SUMS + DIFFERENCES OF CONSTANTS AND CONSTANT MULTIPLES OF VARIABLES.

E.g., $\left\{ \begin{array}{l} 3x = \frac{1}{2}y + 1 \\ y - z = 0 \\ x + 3w = \frac{1}{5} - y \end{array} \right\}$ IS A LINEAR SYSTEM IN x, y, z, w

→ +, -, CONSTANT MULTIPLES ALLOWED FOR VARIABLES

WHILE $\left\{ \begin{array}{l} x^2 + 4 = z \\ \sin y = 0 \\ x + y = 2 \end{array} \right\}$ IS NOT.

→ MULTIPLICATION, DIVISION, AND OTHER FUNCTIONS ARE NOT!

2. VALID OPERATIONS FOR LINEAR SYSTEMS:

- SCALE AN EQUATION BY SOME NONZERO CONSTANT
- ADD A MULTIPLE OF ONE EQUATION TO ANOTHER
- (INTRODUCE OR REMOVE THE EQUATION "0=0") } VALID, BUT NOT PARTICULARLY USEFUL!
- NOTE THAT REARRANGING THE TERMS WITHIN AN EQUATION AND REORDERING THE LIST ARE VALID BUT TRIVIAL OPERATIONS, TOO.
- "SOLVING + SUBSTITUTING" IS ALSO VALID, BUT IT'S ACTUALLY SUBSUMED BY THE OPERATION OF ADDING MULTIPLES (!)

* KEY FEATURE: EACH OF THESE OPERATIONS IS REVERSIBLE
 - I.E., EACH CAN BE UNDONE VIA A SIMILAR OPERATION.
 (YOU CAN'T UNDO MULTIPLICATION BY ZERO...)

3. A LINEAR SYSTEM IS HOMOGENEOUS IF, AFTER MOVING ALL VARIABLES TO THE LEFT AND ALL CONSTANTS TO THE RIGHT*, THE RIGHT-HAND SIDES OF ALL EQUATIONS ARE ZERO:

$$\begin{array}{l} 2y = 4x + z \\ x + 1 = y + 2 - z - 1 \\ 3 + z + 2x = 3 \end{array} \rightsquigarrow \begin{array}{l} -4x + 2y - z = 0 \\ x - y + z = 0 \\ 2x + z = 0 \end{array} \rightarrow \text{HOMOGENEOUS}$$

OTHERWISE, IT'S INHOMOGENEOUS:

$$\begin{array}{l} 2y = 4x + z \\ x = y + 2 - z - 1 \\ 3 + z + 2x = 3 \end{array} \rightsquigarrow \begin{array}{l} -4x + 2y - z = 0 \\ x - y + z = 1 \\ 2x + z = 0 \end{array} \rightarrow \text{INHOMOGENEOUS!}$$

* PUTTING THE EQUATIONS INTO THIS "STANDARD FORM" FIRST IS IMPORTANT, BECAUSE WE MUST ALLOW CONSTANT TERMS TO COMBINE AND, POSSIBLY, CANCEL!

4. (a) A LINEAR SYSTEM CAN BE SOLVED BY "PIVOTING" AND ELIMINATION OF VARIABLES, ITERATIVELY, AS FOLLOWS:

- (PIVOT) • CHOOSE A PIVOT, I.E., A VARIABLE APPEARING IN SOME EQUATION.
- (SCALE) • IF NECESSARY, SCALE THAT EQUATION SO THAT THE COEFFICIENT OF THE PIVOT BECOMES "1." (JUST FOR CONVENIENCE)
- (ELIMINATE) • ADD APPROPRIATE MULTIPLES OF YOUR PIVOT EQUATION TO ELIMINATE THE PIVOT VARIABLE FROM ALL OTHER EQUATIONS.
- (REPEAT) • MARK THE VARIABLE + EQUATION AS "USED", AND REPEAT THE PROCESS UNTIL NO MORE PIVOTS CAN BE FOUND.

(b) ANY REMAINING NON-PIVOT VARIABLES ARE CALLED FREE VARIABLES.

(c) WHEN FINISHED, THERE ARE THREE POSSIBILITIES:

⊖ AN IMPOSSIBLE EQUATION REMAINS, AND THE SYSTEM HAS NO SOLUTIONS (I.E., ITS SOLUTION SET IS EMPTY) (SUCH AS $0=1$)

OTHERWISE, EITHER:

⊕ THERE ARE NO FREE VARIABLES, AND OUR PIVOT EQUATIONS TELL US THE SYSTEM'S UNIQUE SOLUTION.

OR ⊖ AT LEAST ONE FREE VARIABLE REMAINS, IN WHICH CASE WE ARE FREE TO SET THE FREE VARIABLES TO ANY VALUES WE CHOOSE, THEN USE OUR EQUATIONS TO SOLVE FOR THE REST OF THE VARIABLES IN TERMS OF THEM... RESULTING IN AN INFINITE NUMBER OF SOLUTIONS.

5. A LINEAR SYSTEM HAVING NO SOLUTIONS IS CALLED INCONSISTENT;
A LINEAR SYSTEM HAVING ONE OR MORE SOLUTIONS IS CALLED CONSISTENT.

* HOMOGENEOUS SYSTEMS ARE ALWAYS CONSISTENT, BECAUSE WE CAN OBTAIN AT LEAST ONE SOLUTION BY SETTING ALL VARIABLES TO ZERO!

6.
$$\begin{array}{l} X=Y \\ Y=Z \\ Z=X+1 \end{array} \xrightarrow{\text{STANDARD FORM}} \begin{array}{l} X-Y=0 \\ Y-Z=0 \\ Z-X=1 \end{array}$$
 INHOMOGENEOUS

$$\begin{array}{l} X-Y=0 \\ Y-Z=0 \\ Z-X=1 \end{array} \xrightarrow{+1} \begin{array}{l} X-Y=0 \\ Y-Z=0 \\ Z-Y=1 \end{array} \xrightarrow{+1} \begin{array}{l} X-Z=0 \\ Y-Z=0 \\ 0=1 \end{array}$$
 INCONSISTENT (NO SOLUTIONS)

PIVOT ON X IN FIRST EQ. PIVOT ON Y IN SECOND EQ.

7.
$$\begin{array}{l} X=5+Y+Z \\ Y=X+Z-2 \\ Z=3X-1 \end{array} \xrightarrow{\text{STANDARD FORM}} \begin{array}{l} X-Y-Z=5 \\ -X+Y-Z=-2 \\ -3X+Z=-1 \end{array}$$
 INHOMOGENEOUS

$$\begin{array}{l} X-Y-Z=5 \\ -X+Y-Z=-2 \\ -3X+Z=-1 \end{array} \xrightarrow{+1} \begin{array}{l} X-Y-Z=5 \\ -2Z=3 \\ -3Y-2Z=14 \end{array} \xrightarrow{+1} \begin{array}{l} X-Y-Z=5 \\ Z=-\frac{3}{2} \\ -3Y-2Z=14 \end{array}$$

PIVOT ON X IN FIRST EQ. SCALE SECOND EQUATION BY $-\frac{1}{2}$ PIVOT ON Z IN SECOND EQ.

$$\begin{array}{l} X-Y=\frac{7}{2} \\ Z=-\frac{3}{2} \\ -3Y=11 \end{array} \xrightarrow{+1} \begin{array}{l} X-Y=\frac{7}{2} \\ Z=-\frac{3}{2} \\ Y=-\frac{11}{3} \end{array} \xrightarrow{+1} \begin{array}{l} X=-\frac{1}{6} \\ Z=-\frac{3}{2} \\ Y=-\frac{11}{3} \end{array}$$
 CONSISTENT

SCALE THIRD EQUATION BY $-\frac{1}{3}$ PIVOT ON Y IN THIRD EQ. UNIQUE SOLUTION (NO FREE VARIABLES)

8. $x+y=3z+w$
 $2x-z=y$
 $x+z=10w$

STANDARD FORM

$x+y-3z-w=0$
 $2x-y-z=0$
 $x+z-10w=0$

HOMOGENEOUS

$x+y-3z-w=0$
 $-2 \rightarrow 2x-y-z=0$
 $-1 \rightarrow x+z-10w=0$

PIVOT ON x IN FIRST EQ.

$x+y-3z-w=0$
 $-3y+5z+2w=0$
 $-y+4z-9w=0$

SCALE SECOND EQ.
 BY $-\frac{1}{3}$

$x+y-3z-w=0$
 $-1 \rightarrow y-\frac{5}{3}z-\frac{2}{3}w=0$
 $+1 \rightarrow -y+4z-9w=0$

PIVOT ON y IN SECOND EQ.

$x-\frac{4}{3}z-\frac{1}{3}w=0$
 $y-\frac{5}{3}z-\frac{2}{3}w=0$
 $\frac{7}{3}z-\frac{29}{3}w=0$

SCALE THIRD EQ.
 BY $\frac{3}{7}$

$x-\frac{4}{3}z-\frac{1}{3}w=0$
 $y-\frac{5}{3}z-\frac{2}{3}w=0$
 $z-\frac{29}{7}w=0$

PIVOT ON z IN THIRD EQ.

$x-\frac{41}{7}w=0$
 $y-\frac{53}{7}w=0$
 $z-\frac{29}{7}w=0$

CONSISTENT

FREE VARIABLE w;
 $x=\frac{41}{7}w, y=\frac{53}{7}w, z=\frac{29}{7}w.$

9. $3y=11+x+17w$
 $x+y+12z+5w=45$
 $8z+13y=67w+3x+73$
 $16+42w=7y-2x-2z$

STANDARD FORM

$-x+3y-17w=11$
 $x+y+12z+5w=45$
 $-3x+13y+8z-67w=73$
 $2x-7y+2z+42w=-16$

INHOMOGENEOUS

$-x+3y-17w=11$
 $+17 \rightarrow x+y+12z+5w=45$
 $+36 \rightarrow -3x+13y+8z-67w=73$
 $-2 \rightarrow 2x-7y+2z+42w=-16$

PIVOT ON x IN SECOND EQ.

$4y+12z-12w=56$
 $x+y+12z+5w=45$
 $16y+44z-52w=208$
 $-9y-22z+32w=-106$

SCALE FIRST EQUATION
 BY $\frac{1}{7}$

$y+3z-3w=14$
 $+9 \rightarrow x+y+12z+5w=45$
 $-16 \rightarrow 16y+44z-52w=208$
 $-9 \rightarrow -9y-22z+32w=-106$

PIVOT ON y IN FIRST EQ.

$y+3z-3w=14$
 $x+9z+8w=31$
 $-4z-4w=-16$
 $5z+5w=20$

SCALE THIRD EQUATION
 BY $-\frac{1}{7}$

$y+3z-3w=14$
 $x+9z+8w=31$
 $z+w=4$
 $-5 \rightarrow 5z+5w=20$

PIVOT ON z IN THIRD EQ.

$y-6w=2$
 $x-w=-5$
 $z+w=4$
 $0=0$

CONSISTENT

FREE VARIABLE: w

$y=2+6w$
 $x=-5+w$
 $z=4-w$