1. LINEAR SYSTEM (IN STANDARD FORM) → AUGMENTED MATRIX

$$\begin{aligned} -x + 4y - 3z &= 10 \\ 2x \qquad + 5z &= -2 \\ -2y + z &= 0 \end{aligned} \rightsquigarrow \begin{array}{c|c} & \begin{array}{cccc} x & y & z & \text{const.} \end{array} \\ \begin{array}{c} \text{Eq. 1} \\ \text{Eq. 2} \\ \text{Eq. 3} \end{array} & \left[\begin{array}{ccc|c} -1 & 4 & -3 & 10 \\ 2 & 0 & 5 & -2 \\ 0 & -2 & 1 & 0 \end{array}\right] \end{array}$$

- EACH ROW OF THE AUGMENTED MATRIX CORRESPONDS TO AN EQUATION OF THE LINEAR SYSTEM.
- THE PRE-FINAL COLUMNS OF THE AUGMENTED MATRIX CORRESPOND TO THE VARIABLES OF THE SYSTEM, STORING THE COEFFICIENT OF EACH VARIABLE IN THE EQUATION.*
- THE FINAL COLUMN OF THE AUGMENTED MATRIX STORES THE CONSTANTS FOR THE EQUATIONS* IN ITS ROWS.

* AS WRITTEN IN STANDARD FORM: VARIABLES ON THE LEFT, CONSTANTS ON THE RIGHT!

- WHEN TRANSLATING A LINEAR SYSTEM INTO AUGMENTED MATRIX FORM, WE SHOULD BE PARTICULARLY CAREFUL THAT THE SYSTEM IS IN STANDARD FORM AND THAT THE COEFFICIENTS ARE PUT IN THE APPROPRIATE COLUMNS.
  - — BE CAREFUL TO INCLUDE ANY "–"'S!
  - — ANY VARIABLE NOT APPEARING IN AN EQUATION IMPLICITLY HAS A COEFFICIENT OF ZERO!

2. REDUCING AN AUGMENTED MATRIX

(a) OUR THREE BASIC OPERATIONS ARE:
1. SCALE A ROW BY SOME NONZERO FACTOR
2. ADD A MULTIPLE OF ONE ROW TO ANOTHER
3. EXCHANGE TWO ROWS

TECHNICALLY, THIS CAN BE ACHIEVED THROUGH SCALINGS + MULTIPLE ADDITIONS — IF YOU'RE CURIOUS, TRY TO FIGURE OUT HOW!

* WHEN PERFORMING THESE OPERATIONS, KEEP IN MIND THAT THEY ARE APPLIED CONSISTENTLY ACROSS THE ENTIRE ROWS!

(b) A PIVOT IN AN AUGMENTED MATRIX IS THE FIRST NONZERO ENTRY IN A ROW, IF ONE EXISTS TO THE LEFT OF THE LINE. OUR GOAL IN AUGMENTED MATRIX REDUCTION IS FOR EACH PIVOT TO BE THE ONLY NONZERO ENTRY IN ITS COLUMN.

(c) AS WE WORK LEFT-TO-RIGHT THROUGH THE COLUMNS OF THE MATRIX, OUR COLUMN GIVES US A PIVOT IF IT HAS A NONZERO ENTRY IN ANY NON-PIVOT ROW.

IN THIS CASE, WE:
- MOVE THIS ROW TO THE TOP OF THE NON-PIVOT ROWS VIA A ROW-EXCHANGE (IF NECESSARY);
- SCALE THAT ROW SO THAT OUR NEW PIVOT BECOMES A ONE; AND
- KILL ALL NONZERO ENTRIES ABOVE AND BELOW OUR NEW PIVOT, BY ADDING APPROPRIATE MULTIPLES OF OUR ROW.

(THEN WE MOVE ON TO THE NEXT COLUMN, CONTINUING UNTIL WE HIT THE LINE)

(d) WHEN WE'VE FINISHED, IF WE LOCATE THE PIVOTS OF OUR MATRIX, WE SHOULD SEE THEM MOVING DOWNWARD AND RIGHTWARD, EACH PIVOT'S COLUMN CONTAINING ALL ZEROS EXCEPT FOR A ONE AT THE PIVOT.

3. SOLVING LINEAR SYSTEMS BY AUGMENTED MATRIX REDUCTION:

(a) CONSISTENCY IS DETERMINED BY LOOKING AT THE ROWS (IF ANY) HAVING ALL ZEROS TO THE LEFT OF THE LINE: IF ANY HAVE A NONZERO ENTRY TO THE RIGHT, THE SYSTEM IS INCONSISTENT. IF ALL SUCH ROWS HAVE A ZERO TO THE RIGHT (IN PARTICULAR, IF THERE ARE NO SUCH ROWS TO BEGIN WITH), THE SYSTEM IS CONSISTENT.

* IF THE SYSTEM IS INCONSISTENT, WE'RE FINISHED!

(b) THE <u>PIVOT VARIABLES</u> OF THE SYSTEM ARE THOSE CORRESPONDING TO THE COLUMNS THAT CONTAIN PIVOTS; THE REST OF THE VARIABLES (THOSE WHOSE COLUMNS DON'T CONTAIN PIVOTS) ARE <u>FREE VARIABLES</u>.

IF THE SYSTEM IS CONSISTENT:

- IT HAS A UNIQUE SOLUTION IF, AND ONLY IF, THERE ARE NO FREE VARIABLES.
- IT HAS INFINITELY MANY SOLUTIONS IF, AND ONLY IF, THERE IS AT LEAST ONE FREE VARIABLE.

(c) ONCE WE'VE FOUND THE PIVOT AND FREE VARIABLES OF A SYSTEM, ALL THAT WE NEED TO DO IS RE-READ EACH PIVOT ROW AND SOLVE FOR THE PIVOT VARIABLES IN TERMS OF THE FREE VARIABLES (IF ANY).

4. (a) $\begin{aligned} x - y - z &= 5 \\ -x + y - z &= -2 \\ -3x \phantom{{}+y} + z &= -1 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ -1 & 1 & -1 & -2 \\ -3 & 0 & 1 & -1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & -\frac{11}{3} \\ 0 & 0 & 1 & -3/2 \end{array}\right] \rightsquigarrow \begin{aligned} x &= -\tfrac{1}{6} \\ y &= -\tfrac{11}{3} \\ z &= -\tfrac{3}{2} \end{aligned}$ (CONSISTENT)

(d) $\begin{aligned} 5x + 10y + 35z &= 0 \\ -3x - 7y + 24z &= 0 \\ -4x - 9y + 31z &= 0 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 5 & 10 & 35 & 0 \\ -3 & -7 & 24 & 0 \\ -4 & -9 & 31 & 0 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$ (CONSISTENT)

(b) $\begin{aligned} x + y - 3z - w &= 0 \\ 2x - y - z \phantom{{}-10w} &= 0 \\ x \phantom{{}+y} + z - 10w &= 0 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -3 & -1 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -10 & 0 \end{array}\right]$

$\rightsquigarrow \begin{array}{c} \begin{array}{ccccc} & & & \text{FREE} & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -41/7 & 0 \\ 0 & 1 & 0 & -53/7 & 0 \\ 0 & 0 & 1 & -29/7 & 0 \end{array}\right] \end{array} \rightsquigarrow$

FREE VARIABLE: w

$x = \frac{41}{7} w$

$y = \frac{53}{7} w$ (CONSISTENT)

$z = \frac{29}{7} w$

(e) $\begin{aligned} 2a + 8b &= 20 \\ 5a + 25b &= 55 \\ 3a + 11b &= 29 \\ 3a + 13b &= 31 \\ 6a + 19b &= 55 \end{aligned} \rightsquigarrow \left[\begin{array}{cc|c} 2 & 8 & 20 \\ 5 & 25 & 55 \\ 3 & 11 & 29 \\ 3 & 13 & 31 \\ 6 & 19 & 55 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{aligned} a &= 6 \\ b &= 1 \end{aligned}$ (CONSISTENT)

(c) $\begin{aligned} -x + 3y \phantom{{}+12z} - 17w &= 11 \\ x + y + 12z + 5w &= 45 \\ -3x + 13y + 8z - 67w &= 73 \\ 2x - 7y + 2z + 42w &= -16 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} -1 & 3 & 0 & -17 & 11 \\ 1 & 1 & 12 & 5 & 45 \\ -3 & 13 & 8 & -67 & 73 \\ 2 & -7 & 2 & 42 & -16 \end{array}\right]$

$\rightsquigarrow \begin{array}{c} \begin{array}{ccccc} & & & \text{FREE} & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -5 \\ 0 & 1 & 0 & -6 & 2 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \rightsquigarrow$

FREE VARIABLE: w

$x = -5 + w$

$y = 2 + 6w$ (CONSISTENT)

$z = 4 - w$

(f) $\begin{aligned} 4x - y + 4z - u - 5v &= -6 \\ -20x + 2y - 17z + 5u + 25v &= 21 \\ -24x + 9y - 24z + 6u + 30v &= 40 \\ 12x - 4y + 13z - 3u - 15v &= -21 \end{aligned}$

$\rightsquigarrow \left[\begin{array}{ccccc|c} 4 & -1 & 4 & -1 & -5 & -6 \\ -20 & 2 & -17 & 5 & 25 & 21 \\ -24 & 9 & -24 & 6 & 30 & 40 \\ 12 & -4 & 13 & -3 & -15 & -21 \end{array}\right]$

$\rightsquigarrow \begin{array}{c} \begin{array}{cccccc} & & & \text{FREE} & \text{FREE} & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{4} & -\frac{5}{4} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \rightsquigarrow$

FREE VARIABLES: u, v

$x = \frac{1}{2} + \frac{1}{4}u + \frac{5}{4}v$

$y = \frac{4}{3}$

$z = -\frac{5}{3}$ (CONSISTENT)