1. WHEN ROW-REDUCING AN AUGMENTED MATRIX, WE NOTE THAT WHILE THE ROWS OF THE MATRIX GET MIXED TOGETHER, ALL OF OUR OPERATIONS PRESERVE THE COLUMNS — THUS, SPLITTING THE MATRIX INTO COLUMNS IS A QUITE SENSIBLE IDEA.

   ↳ NOTE THAT WE'VE TURNED OUR WHOLE PROBLEM AROUND! ORIGINALLY, WE GROUPED OUR INFORMATION INTO EQUATIONS (ROWS), BUT VIEWING IT THE OTHER WAY AROUND LEADS US DIRECTLY INTO THE REALM OF LINEAR ALGEBRA!

2. A COLUMN VECTOR (OF SOME SIZE) IS SIMPLY A COLLECTION OF NUMBERS (ITS ENTRIES) ARRANGED IN ORDER INTO A COLUMN. A ZERO VECTOR IS ONE WHOSE ENTRIES ARE ALL ZEROS.

3. THE FUNDAMENTAL ALGEBRAIC OPERATION ON COLUMN VECTORS IS THE LINEAR COMBINATION, E.G.,

$$3\begin{bmatrix}2\\0\\-1\end{bmatrix} + 2\begin{bmatrix}0\\-2\\4\end{bmatrix} + 0\begin{bmatrix}3\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}5\\3\\5\end{bmatrix} \quad \text{MULTIPLY}$$

$$= \begin{bmatrix}6\\0\\-3\end{bmatrix} + \begin{bmatrix}0\\-4\\8\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-5\\-3\\-5\end{bmatrix} \quad \text{THEN ADD UP}$$

$$= \begin{bmatrix}1\\-7\\0\end{bmatrix}$$

GIVEN COEFFICIENTS FOR SOME COLLECTION OF VECTORS, SCALE THE ENTRIES OF EACH VECTOR BY ITS COEFFICIENT AND ADD UP CORRESPONDING ENTRIES — THE RESULT IS ANOTHER VECTOR!

- THIS OPERATION SUBSUMES BOTH SCALING AND ADDITION:

$$3\begin{bmatrix}2\\1\\5\end{bmatrix} \quad \text{IS A LINEAR COMBINATION OF ONE VECTOR;}$$

$$\begin{bmatrix}4\\-1\\3\end{bmatrix} + \begin{bmatrix}5\\5\\0\end{bmatrix} = 1\begin{bmatrix}4\\-1\\3\end{bmatrix} + 1\begin{bmatrix}5\\5\\0\end{bmatrix} \quad \text{IS A LINEAR COMBINATION OF TWO!}$$

- CONVERSELY, ANY LINEAR COMBINATION CAN BE BROKEN DOWN INTO SOME COMBINATION OF SCALINGS + SUMS!

4. WHEN REINTERPRETED AS A SINGLE EQUATION OF VECTORS, A LINEAR SYSTEM POSES THE QUESTION OF WHETHER SOME TARGET VECTOR (THE RIGHT-HAND SIDE) CAN BE WRITTEN AS A LINEAR COMBINATION OF SOME GIVEN COLUMN VECTORS (ON THE LEFT-HAND SIDE) — AND, IF SO, HOW? E.G.,

CAN THIS TARGET VECTOR...

$$\begin{aligned} 2a + 8b - 2c &= -26 \\ 4a + 13b - 10c &= -25 \\ -2a - 5b + 7c &= 2 \\ 6a + 30b + 10c &= -144 \end{aligned} \quad \rightsquigarrow \quad a\begin{bmatrix}2\\4\\-2\\6\end{bmatrix} + b\begin{bmatrix}8\\13\\-5\\30\end{bmatrix} + c\begin{bmatrix}-2\\-10\\7\\10\end{bmatrix} = \begin{bmatrix}-26\\-25\\2\\-144\end{bmatrix}$$

...BE WRITTEN AS A LINEAR COMBINATION OF THESE GIVEN COLUMN VECTORS?

- IN THE CASE OF A HOMOGENEOUS SYSTEM, THE QUESTION BECOMES SIMPLY ONE OF HOW THE ZERO VECTOR CAN BE WRITTEN AS A LINEAR COMBINATION OF THE GIVEN VECTORS.

5. ONCE WE'VE SOLVED A SYSTEM, WE CAN COLLECT THE VARIABLES IN ITS SOLUTION INTO COLUMN VECTORS, AS WELL — E.G.:

u, v: FREE VARIABLES

$$\begin{aligned} x &= 1-u+v \\ y &= 3u+4v \\ z &= -2+u-v \end{aligned} \quad \rightsquigarrow \quad \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix} = \begin{bmatrix} 1-u+v \\ 3u+4v \\ -2+u-v \\ u \\ v \end{bmatrix}$$

WE CAN THEN SPLIT OFF THE FREE VARIABLES' CONTRIBUTIONS BY SPLITTING THE VECTOR INTO ITS CONSTANT PART AND EACH VARIABLE'S PART AND FACTORING OUT THE VARIABLES:

$$\begin{bmatrix} 1-u+v \\ 3u+4v \\ -2+u-v \\ u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -u \\ 3u \\ u \\ u \\ 0 \end{bmatrix} + \begin{bmatrix} v \\ 4v \\ -v \\ 0 \\ v \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

* THUS, BOTH THE PROBLEM POSED BY A LINEAR SYSTEM AND ITS SOLUTION NATURALLY FALL INTO EXPRESSION VIA LINEAR COMBINATIONS OF COLUMN VECTORS!

6. CONSIDERING OUR METHOD FOR SOLVING A LINEAR SYSTEM, WE SEE THAT THE PARTICULAR TARGET VECTOR IN OUR SYSTEM ONLY AFFECTS THE SOLUTION'S CONSTANT SUMMAND — IN THE HOMOGENEOUS CASE, THE CONSTANT SUMMAND WILL BE ZERO AND CAN BE LEFT OUT.

7. (a) $$-2\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -2+0+3 \\ -8-3+0 \\ -10-3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ -9 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+1-0 \\ 1+1-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(c) $$8\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} - 5\begin{bmatrix} 3 \\ 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 8-15 \\ 0-10 \\ -8-0 \\ -16+15 \end{bmatrix} = \begin{bmatrix} -7 \\ -10 \\ -8 \\ -1 \end{bmatrix}$$

8. $$a\begin{bmatrix} -4 \\ 4 \\ -3 \end{bmatrix} + b\begin{bmatrix} 6 \\ 30 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 5 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{cc|c} -4 & 6 & 5 \\ 4 & 30 & 13 \\ -3 & 7 & 5 \end{array}\right] \end{array}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightsquigarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

YES — THE ONLY POSSIBLE COEFFICIENT VECTOR IS $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

9. $$a\begin{bmatrix} 4 \\ -20 \\ 40 \end{bmatrix} + b\begin{bmatrix} 7 \\ -51 \\ 67 \end{bmatrix} + c\begin{bmatrix} -1 \\ 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 8 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 4 & 7 & -1 & 2 \\ -20 & -51 & 5 & 10 \\ 40 & 67 & -10 & 8 \end{array}\right] \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & \frac{43}{16} \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & 0 & -\frac{63}{4} \end{array}\right] \rightarrow \text{INCONSISTENT!!}$$

NO — IT CAN'T BE WRITTEN AS SUCH A LINEAR COMBINATION (THE SYSTEM IS INCONSISTENT, I.E., IT HAS NO SOLUTIONS)

10. (a)

CAN THIS ...

$$x\begin{bmatrix}1\\-1\\-3\end{bmatrix}+y\begin{bmatrix}-1\\1\\0\end{bmatrix}+z\begin{bmatrix}-1\\-1\\1\end{bmatrix}=\begin{bmatrix}5\\-2\\-1\end{bmatrix}$$

... BE WRITTEN AS A L.C. OF THESE?

YES, FOR $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1/6\\-11/3\\-3/2\end{bmatrix}$

(b)

CAN THE ZERO VECTOR ...

$$x\begin{bmatrix}1\\2\\1\end{bmatrix}+y\begin{bmatrix}1\\-1\\0\end{bmatrix}+z\begin{bmatrix}-3\\-1\\1\end{bmatrix}+w\begin{bmatrix}-1\\0\\-10\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

... BE WRITTEN AS A L.C. OF THESE?

YES, FOR $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=w\begin{bmatrix}41/7\\53/7\\29/7\\1\end{bmatrix}$

(c)

CAN THIS ...

$$x\begin{bmatrix}-1\\1\\-3\\2\end{bmatrix}+y\begin{bmatrix}3\\1\\13\\-7\end{bmatrix}+z\begin{bmatrix}0\\12\\8\\2\end{bmatrix}+w\begin{bmatrix}-17\\5\\-67\\42\end{bmatrix}=\begin{bmatrix}11\\45\\73\\-16\end{bmatrix}$$

... BE WRITTEN AS A L.C. OF THESE?

YES, FOR $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-5\\2\\4\\0\end{bmatrix}+w\begin{bmatrix}1\\6\\-1\\1\end{bmatrix}$

(d)

CAN THE ZERO VECTOR ...

$$x\begin{bmatrix}5\\-3\\-4\end{bmatrix}+y\begin{bmatrix}10\\-7\\-9\end{bmatrix}+z\begin{bmatrix}35\\24\\31\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

... BE WRITTEN AS A L.C. OF THESE? YES, FOR $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(YES, BUT ONLY IN THE "TRIVIAL" WAY!)

(e)

CAN THIS ...

$$a\begin{bmatrix}2\\5\\3\\3\\6\end{bmatrix}+b\begin{bmatrix}8\\25\\11\\13\\19\end{bmatrix}=\begin{bmatrix}20\\55\\29\\31\\55\end{bmatrix}$$

.. BE WRITTEN AS A L.C. OF THESE? YES, FOR $\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}6\\1\end{bmatrix}$

(f)

CAN THIS VECTOR ...

$$x\begin{bmatrix}4\\-20\\-24\\12\end{bmatrix}+y\begin{bmatrix}-1\\2\\9\\-4\end{bmatrix}+z\begin{bmatrix}4\\-17\\-24\\13\end{bmatrix}+u\begin{bmatrix}-1\\5\\6\\-3\end{bmatrix}+v\begin{bmatrix}-5\\25\\30\\-15\end{bmatrix}=\begin{bmatrix}-6\\21\\40\\-21\end{bmatrix}$$

... BE WRITTEN AS A LINEAR COMBINATION OF THESE?

YES, FOR $\begin{bmatrix}x\\y\\z\\u\\v\end{bmatrix}=\begin{bmatrix}1/2\\4/3\\-5/3\\0\\0\end{bmatrix}+u\begin{bmatrix}1/4\\0\\0\\1\\0\end{bmatrix}+v\begin{bmatrix}5/4\\0\\0\\0\\1\end{bmatrix}$

11. WE'RE LOOKING AT $a\begin{bmatrix}1\\8\end{bmatrix}+b\begin{bmatrix}2\\7\end{bmatrix}+c\begin{bmatrix}-1\\1\end{bmatrix}+d\begin{bmatrix}0\\-9\end{bmatrix}=?$,

I.E., $\left[\begin{array}{cccc|c}1&2&-1&0&?\\8&7&1&-9&?\end{array}\right]$, FOR VARIOUS TARGETS.

(a) $$\left[\begin{array}{cccc|c}1&2&-1&0&19\\8&7&1&-9&107\end{array}\right] \rightsquigarrow \begin{array}{c}\begin{array}{ccccc}&&\text{FREE}&\text{FREE}&\end{array}\\ \left[\begin{array}{cccc|c}1&0&1&-2&9\\0&1&-1&1&5\end{array}\right]\end{array}$$

$$\rightsquigarrow \begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}9-c+2d\\5+c-d\\c\\d\end{bmatrix}=\begin{bmatrix}9\\5\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+d\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

(b) $$\left[\begin{array}{cccc|c}1&2&-1&0&2/5\\8&7&1&-9&5\end{array}\right] \rightsquigarrow \begin{array}{c}\begin{array}{ccccc}&&\text{FREE}&\text{FREE}&\end{array}\\ \left[\begin{array}{cccc|c}1&0&1&-2&4/5\\0&1&-1&1&-1/5\end{array}\right]\end{array}$$

$$\rightsquigarrow \begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}4/5-c+2d\\-1/5+c-d\\c\\d\end{bmatrix}=\begin{bmatrix}4/5\\-1/5\\0\\0\end{bmatrix}+c\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+d\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

(c) $$\left[\begin{array}{cccc|c}1&2&-1&0&0\\8&7&1&-9&0\end{array}\right] \rightsquigarrow \begin{array}{c}\begin{array}{ccccc}&&\text{FREE}&\text{FREE}&\end{array}\\ \left[\begin{array}{cccc|c}1&0&1&-2&0\\0&1&-1&1&0\end{array}\right]\end{array}$$

$$\rightsquigarrow \begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}-c+2d\\c-d\\c\\d\end{bmatrix}=c\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+d\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

(d) THE "FREE PARTS" OF THESE ARE ALL THE SAME
— THE SOLUTIONS DIFFER ONLY IN THEIR "CONSTANT" VECTORS!