Span, linear independence, and basis

Suppose that $\mathscr{C} = \{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_n \,\}$ is a collection of vectors in some vector space V.

The span

- Definition: $\operatorname{span}\mathscr{C} = \{\, \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \;:\; \alpha_1, \alpha_2, \ldots, \alpha_n \text{ scalars} \,\}$ —i.e., the set of all linear combinations of $\mathscr{C}$.
  - Logical unraveling: $\boxed{\vec{v} \in \operatorname{span}\mathscr{C} \text{ means } \exists \text{ scalars } \alpha_1, \alpha_2, \ldots, \alpha_n \text{ such that } \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n}$.
  - For any collection $\mathscr{C}$ in V, $\operatorname{span}\mathscr{C}$ is a *subspace* of V (this can be proven directly via our three-point checklist for subspaces).
  - Conceptually, if we pick a collection $\mathscr{C}$ of vectors from V, taking the span of $\mathscr{C}$ "grows" it into a subspace of V.
  - $\mathscr{C}$ *spans* V if $\operatorname{span}\mathscr{C} = V$; conceptually, this means that when we "grow" $\mathscr{C}$, the resulting subspace "fills" all of V (see logical unraveling below).
- In the case that V consists of column vectors:
  - Checking whether $\vec{v} \in \operatorname{span}\mathscr{C}$ is equivalent to determining whether the system $[\,\mathscr{C} \mid \vec{v}\,]$ is consistent.
  - Similarly, checking whether $\operatorname{span}\mathscr{C} \supset \operatorname{span}\mathscr{D}$ is equivalent to checking the *multiply*-augmented system $[\,\mathscr{C} \mid \mathscr{D}\,]$ for consistency in all augmented columns.
  - To check whether $\operatorname{span}\mathscr{C} = \operatorname{span}\mathscr{D}$, check *both* $[\,\mathscr{C} \mid \mathscr{D}\,]$ and $[\,\mathscr{D} \mid \mathscr{C}\,]$ for consistency.

Linear relations, dependence, and independence

- A *linear relation* on $\mathscr{C}$ is a way of writing $\vec{0}$ as a linear combination of $\mathscr{C}$; it is called *trivial* if every coefficient is zero, or *nontrivial* if at least one coefficient is nonzero.
  - $\mathscr{C}$ is called *linearly dependent* if there is a nontrivial linear relation on $\mathscr{C}$, i.e., $\boxed{\exists \text{ scalars } \alpha_1, \alpha_2, \ldots, \alpha_n, \textit{ not all zero}, \text{ such that } \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n = \vec{0}}$.
  - $\mathscr{C}$ is called *linearly independent* if there is no nontrivial linear relation on $\mathscr{C}$, i.e., every linear relation on $\mathscr{C}$ is trivial (see below).

	Spanning sets	Bases	Linearly independent sets
Logical unraveling	$\mathscr{C}$ **spans** V means $\forall\, \vec{v} \in V,\ \exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n$	$\mathscr{C}$ is a **basis** for V means $\mathscr{C}$ spans V and is linearly independent in V	$\mathscr{C}$ is **linearly independent** in V means $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0} \Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_n = 0$
Vectors in V	Every $\vec{v} \in V$ can be written as a l.c. of $\mathscr{C}$ in *at least* one way.	Every $\vec{v} \in V$ can be written as a l.c. of $\mathscr{C}$ in *exactly* one way.	Every $\vec{v} \in V$ can be written as a l.c. of $\mathscr{C}$ in *at most* one way.
Property-preserving			
…insertions:	*Insert* any l.c. of $\mathscr{C}$.	[*none*]	*Insert* any $\vec{v} \in V$ not in $\operatorname{span}\mathscr{C}$.
…removals:	*Remove* any l.c. of the rest.	[*none*]	*Remove* any vector of $\mathscr{C}$.
…replacements:	[same $\longrightarrow$]	*Replace* a vector of $\mathscr{C}$ with any l.c. of $\mathscr{C}$ having a nonzero coefficient for that vector.	[$\longleftarrow$ same]
Column vectors			
…and linear systems	Each system arising from $\mathscr{C}$ has *at least* one solution.	Each system arising from $\mathscr{C}$ has *exactly* one solution.	Each system arising from $\mathscr{C}$ has *at most* one solution.
…checking	Does $[\mathscr{C}]$ give a pivot in every *row*?	Does $[\mathscr{C}]$ give a pivot in every row *and* every column?	Does $[\mathscr{C}]$ give a pivot in every *column*?