Basic Logic

We can build logical expressions from simpler ones via logical predicates and quantifiers, just as we combine numbers by arithmetic operations—but instead of numbers, our building blocks are logical ***propositions*** (statements of truth that are either *true* or *false*), and our operations are logical predicates such as "and", "or", "not", etc.

Predicates

Predicates play the role constants and functions in the realm of logic.

- Predicates take as input some number of propositions and, depending on their values, evaluate to either *true* or *false*.
  - As special cases, we have the *constant* predicates ***true*** and ***false*** themselves.
  - The basic higher-order predicates can be explicitly defined via truth tables:

not P:

P	T	F
not P	F	T

P and Q:

Q \ P	T	F
T	T	F
F	F	F

P or Q:

Q \ P	T	F
T	T	T
F	T	F

$P \Rightarrow Q$:

Q \ P	T	F
T	T	T
F	F	T

$P \Leftrightarrow Q$:

Q \ P	T	F
T	T	F
F	F	T

- Note that "or" is *inclusive*—i.e., (*true* or *true*) is *true*. [We use "xor" for exclusive-or (very rarely).]
- The predicate "$\Rightarrow$" is read "implies"
  - "$P \Rightarrow Q$" means "If P, then Q"; we call P the ***hypothesis*** and Q the ***conclusion*** of the implication.
  - Note that this predicate is not symmetric—e.g., $P \Rightarrow Q$ is not equivalent to $Q \Rightarrow P$.
  - Also, note that *false* $\Rightarrow Q$ for *any* Q—i.e., a false hypothesis allows us to deduce anything we like!
- The predicate "$\Leftrightarrow$" is read "if and only if" (equivalent to "$\Leftarrow$ and $\Rightarrow$"); it serves as our "equal sign" for logical propositions.

Quantifiers

Quantifiers allow us to specify the roles of variables in a proposition:

- The ***universal quantifier*** $\forall$'s syntax is $\boxed{\forall\, x,\ P}$, read "for all x, P."
  This is *true* just when P is *true* for each and every value of x; it is *false* if even one choice of x makes P *false*.
- The ***existential quantifier*** $\exists$'s syntax is $\boxed{\exists\, x \text{ such that } P}$, read "there exists x such that P."
  This is *true* just when there is at least one value of x for which P is *true*; it is *false* if P is *false* for every choice of x.

The values allowed by a quantifier are often restricted implicitly (from context) or explicitly.
- e.g., "$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \ldots$" refers only to positive real numbers ε and δ.

Unquantified variables in an expression are called ***free variables***; by convention, free variables are universally quantified—be aware of free variables and make these quantifications explicit when necessary (particularly when negating a proposition).

Logical algebra

We can manipulate logical expressions just as we manipulate numerical ones, via the following rules:

Associativity	P and (Q and R) $\Leftrightarrow$ (P and Q) and R	P or (Q or R) $\Leftrightarrow$ (P or Q) or R
Commutativity	P and Q $\Leftrightarrow$ Q and P	P or Q $\Leftrightarrow$ Q or P
Distributivity	P and (Q or R) $\Leftrightarrow$ (P and Q) or (P and R)	P or (Q and R) $\Leftrightarrow$ (P or Q) and (P or R)
Units	P and *true* $\Leftrightarrow$ P $\Leftrightarrow$ P or *false*	
Negation	not *true* $\Leftrightarrow$ *false*	not(not P) $\Leftrightarrow$ P
	not(P and Q) $\Leftrightarrow$ (not P) or (not Q)	not(P or Q) $\Leftrightarrow$ (not P) and (not Q)
	not($\forall\, x,\ P$) $\Leftrightarrow$ $\exists\, x$ such that not P	not($\exists\, x$ such that P) $\Leftrightarrow$ $\forall\, x,$ not P
Implication	($P \Rightarrow Q$) $\Leftrightarrow$ (Q or not P)	$P \Rightarrow P$ or Q
	($P \Leftrightarrow Q$) $\Leftrightarrow$ ($P \Rightarrow Q$) and ($Q \Rightarrow P$)	P and $Q \Rightarrow P$
	not($P \Rightarrow Q$) $\Leftrightarrow$ P and not Q	

Other identities	*false* and P $\Leftrightarrow$ *false*	P and P $\Leftrightarrow$ P	P and (not P) $\Leftrightarrow$ *false*
	true or P $\Leftrightarrow$ *true*	P or P $\Leftrightarrow$ P	P or (not P) $\Leftrightarrow$ *true*

Mathematical Proof

Modern mathematics centers on logical propositions (which may be either *true* or *false*) about mathematical concepts. Rather than judging such "truth" subjectively on the basis of faith or intuition, or relying on non-absolute methods such as experimental testing, in mathematics we demand that a proposition be justified by a *proof* in order to be accepted—once proven, a proposition achieves special status in mathematics and is often called a ***theorem*** (usually when it's a final result) or ***lemma*** (when it's a step toward something larger).

Aside: Mathematics is the study of logical relationships between mathematical concepts. Important concepts are given names and formal definitions in order to allow proper study of them, and proofs establish the logical connections between points of interest, providing us pathways of truth through the mathematical landscape. At first, we work with basic definitions and establish short logical pathways; building on this, we then establish theorems that allow us to jump greater logical distances in a single step; in turn, we use these theorems to prove larger theorems that span even greater logical distances, and so on. . . the result is a logical infrastructure in the mathematical landscape that which comprises our "understanding" of mathematics—including logical reasoning skills, concepts, definitions, theorems, and proofs.

Proof

A ***proof*** is a sequence of logical propositions, in which each proposition is logically justified by some combination of axioms, definitions, established theorems, and the propositions *preceding* it in the proof.

- The steps in a proof must build from start to finish.
- Proofs are not just about a proposition *being* true—they're about establishing the necessary connections to *justify* its truth.
  - If each step in a proof is valid, then the entire logical chain of the proof is valid.
  - Any improperly justified step in a proof invalidates the entire proof (even if the proposition itself is true!).
- A few tips to keep in mind when writing a proof:
  - Keep close track of what you already *know* (!) and what you're trying to *show* (?).
  - The first step of many proofs is to use *definitions* of terms to unravel them into the objects and logical propositions that they represent; once this has been done, these objects and propositions can be analyzed and combined to construct a proof.
  - Keep an eye out for any known *results* or *theorems* that relate to the proposition you're trying to prove—when something is already known about the concepts at play, a theorem can allow you to prove your proposition without unraveling definitions—this becomes increasingly important as the concepts being studied increase in complexity.

Proof techniques

Noting how the the structure of the proposition we'd like to prove is built from smaller propositions tells us a great deal about how to prove it:

- Each possible outermost logical construction leads us to a corresponding *direct* line of proof:
  - $\underline{\forall x,\ P}$: · "Let x be given." ← you must take the x that's given to you, not choose one yourself
    · Show that P is *true*.
  - $\underline{\exists x \text{ such that } P}$: [Find an x that makes P *true*] ← this is only scratch work, not part of the proof
    · "Let $x = \ldots$"
    · Show that P is *true* for this x.
  - $\underline{P \Rightarrow Q}$: · "Suppose P."
    · Show that Q is *true*.
  - $\underline{P \Leftrightarrow Q}$: Show $[P \Rightarrow Q \text{ and } Q \Rightarrow P]$ —or— connect P to Q via a chain of $\Leftrightarrow$'s.
- A proposition can also be proven *indirectly* by showing that its logical negation is *false*; the advantage that this sometimes provides is that negation changes the outermost operation of the expression, allowing a different line of attack.
- There are four common means of proving the most common type of mathematical assertion, the *implication* $\underline{P \Rightarrow Q}$:
  - Suppose P and show Q. (*directly*)
  - Suppose $[\text{not } Q]$ and show $[\text{not } P]$. (*by the contrapositive*)
  - Show that $[Q \text{ or not } P]$ is *true*. (*by definition*)
  - Suppose $[P \text{ and not } Q]$ and deduce a *false* proposition. (*by contradiction*)