Linear systems: in principle

1. What is a *linear system*?
   What operations are allowed for variables in a linear system? What operations are not allowed?

2. What are the basic operations that we can perform on a linear system without changing its solution set, and what key feature do they possess (that, for example, scaling an equation by zero doesn't)?

3. What does it mean for a linear system to be *homogeneous* or *inhomogeneous*?
   Why is it important to put a system in "standard form" before determining this?

4. Outline our iterative algorithm for solving a linear system, including:
   (a) What is the general procedure of locating pivots and eliminating variables?
   (b) What do we call the variables that are left after we've eliminated as many as possible?
   (c) When finished, how do we determine the linear system's solution or solutions (if any)?
   How many solutions can a linear system have?

5. What does it mean for a linear system to be *consistent* or *inconsistent*?
   Which type of linear system is guaranteed to be consistent, and why?

. . . and in practice

For each of the following linear systems:

- Put the system in standard form and classify it as homogeneous or inhomogeneous.
- Eliminate variables via pivoting, identify the free variables, and find the solution set of the system.
- Classify the system as consistent or inconsistent.

6. $x = y$
   $y = z$
   $z = x + 1$

7. $x = 5 + y + z$
   $y = x + z - 2$
   $z = 3x - 1$

8. $x + y = 3z + w$
   $2x - z = y$
   $x + z = 10w$

9. $3y = 11 + x + 17w$
   $x + y + 12z + 5w = 45$
   $8z + 13y = 67w + 3x + 73$
   $16 + 42w = 7y - 2x - 2z$

Augmented matrix notation and Gaussian elimination: in principle

1. Describe augmented matrix notation for linear systems:
   - What does an *augmented matrix* look like?
   - What does each row correspond to?
   - What do the pre-final columns correspond to, and what do they store?
   - What special meaning does the final column have?
   - When translating a linear system into matrix form, what should we be particularly careful about?
   - When a variable doesn't appear in an equation, what coefficient does it implicitly have?

2. Consider the iterative process of reducing an augmented matrix by *Gaussian elimination.*
   (a) What are the three basic operations that we perform on an augmented matrix?
   What should we keep in mind as we perform them?

   (b) What is a *pivot*? How many pivots is each column allowed to contain after reduction?

   (c) As we work left-to-right through the columns of the matrix,
   . . . how do we determine if the column gives us a new pivot?
   . . . when we locate a new pivot, what do we do?

   (d) How can we tell when we've finished reducing a matrix?

3. Solving linear systems by augmented matrix reduction—once a matrix has been fully reduced:
   (a) How can we immediately determine whether the system is consistent or inconsistent?

   (b) How do we determine the *pivot variables* and *free variables* of the system?
   What do they tell us about how many solutions the system has?

   (c) How do we find the system's solution set?

. . . and in practice

4. For each of the following linear systems:
   - Translate it into augmented matrix form.
   - Reduce the matrix fully, circle its pivots, and mark its free columns.
   - Classify the system as consistent or inconsistent and, if consistent, find its solution set.

(a) $x = 5 + y + z$
$y = x + z - 2$
$z = 3x - 1$

(b) $x + y = 3z + w$
$2x - z = y$
$x + z = 10w$

(c) $3y = 11 + x + 17w$
$x + y + 12z + 5w = 45$
$8z + 13y = 67w + 3x + 73$
$16 + 42w = 7y - 2x - 2z$

(d) $5x + 10y + 35z = 0$
$-3x - 7y + 24z = 0$
$-4x - 9y + 31z = 0$

(e) $2a + 8b = 20$
$5a + 25b = 55$
$3a + 11b = 29$
$3a + 13b = 31$
$6a + 19b = 55$

(f) $4x + 4z = u + 5v + y - 6$
$2y + 25v + 5u = 21 + 17z + 20x$
$-24x + 9y - 24z + 6u + 30v - 40 = 0$
$21 + 12x + 13z = 3u + 15v + 4y$

Reinterpreting linear systems, part 1: the vector paradigm, in principle

1. Analyzing our matrix approach to solving linear systems, what suggests that we might profit from splitting the matrix into columns and reintepreting?

2. What is a *column vector* (of some size)? What is a *zero vector*?

3! What is the fundamental algebraic operation that we perform on column vectors?
How do we compute it? What basic arithmetic operations does it subsume, and how does it relate to them?

4. In terms of linear combinations of column vectors, what is the equivalent formulation of the question posed by a linear system? What specific question about column vectors is posed by a *homogeneous* system?

5. How can we express a linear system's *solutions* via column vectors, and how do linear combinations allow us to split off the contributions from the free variables?

6. What does expressing solutions as linear combinations of column vectors allow us to observe about the relationship between the solutions of a [consistent] linear system and those of its associated homogeneous system?
Why is this the case?

. . . and in practice

7. Simplify the following linear combinations of column vectors:

(a) $-2\begin{bmatrix}1\\4\\5\end{bmatrix}+3\begin{bmatrix}0\\-1\\-1\end{bmatrix}+\begin{bmatrix}3\\0\\4\end{bmatrix}$ (b) $\begin{bmatrix}2\\1\end{bmatrix}-\begin{bmatrix}-1\\-1\end{bmatrix}-\begin{bmatrix}0\\3\end{bmatrix}$ (c) $8\begin{bmatrix}1\\0\\-1\\-2\end{bmatrix}-5\begin{bmatrix}3\\2\\0\\-3\end{bmatrix}$

8. Can $\begin{bmatrix}5\\13\\5\end{bmatrix}$ be written as a linear combination of $\begin{bmatrix}-4\\4\\-3\end{bmatrix}$ and $\begin{bmatrix}6\\30\\7\end{bmatrix}$?
If so, what are the possible coefficient vectors? If not, why not?

9. Can $\begin{bmatrix}2\\10\\8\end{bmatrix}$ be written as a linear combination of $\begin{bmatrix}4\\-20\\40\end{bmatrix}$, $\begin{bmatrix}7\\-51\\67\end{bmatrix}$, and $\begin{bmatrix}-1\\5\\-10\end{bmatrix}$?
If so, what are the possible coefficient vectors? If not, why not?

10. Rephrase each linear system from the previous problem set in terms of linear combinations of column vectors, and reëxpress your solutions in terms of column vectors, splitting off any contributions from free variables.

11. Consider the collection $\mathscr{C}=\left\{\begin{bmatrix}1\\8\end{bmatrix},\begin{bmatrix}2\\7\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}0\\-9\end{bmatrix}\right\}$ of column vectors.

(a) Determine all ways in which $\begin{bmatrix}19\\107\end{bmatrix}$ can be written as a linear combination of $\mathscr{C}$;
express your solutions as coefficient vectors, splitting off the free variable's contribution.

(b) Do as in part (a), but for the vector $\begin{bmatrix}\frac{2}{5}\\5\end{bmatrix}$.

(c) One more time, for the vector $\begin{bmatrix}0\\0\end{bmatrix}$.

(d) What common feature do your answers to parts (a), (b), and (c) share?

Introduction to logic and proof

1. What do we mean by a *proof* of a logical assertion? What does a proof consist of?

2. What characteristics particular to mathematics make proofs possible?

3. What purpose do proofs serve in mathematics, and why is this important to the discipline?

4. Consider the logical statement "$P \Rightarrow Q$." ["P implies Q," or "If P, then Q"]
   (a) What do we call P and Q in this context?
   (b) How do we directly to prove that such a statement is *true*?
   (c) What is a *counterexample* to an if-then statement?
   What can we conclude from the existence of a counterexample?
   (d) What related statement is logically equivalent to this one? What is it called?

5. Consider the logical statement "$\forall\, x,\ Q$." ["For all x, Q"]
   (a) How do we directly prove that such a statement is *true*?
   (b) What must be shown to demonstrate that such a statement is *false*?

6. Consider the logical statement "$\exists\, x$ such that Q." ["There exists an x for which Q"]
   (a) How do we directly prove that such a statement is *true*?
   (b) What must be shown to demonstrate that such a statement is *false*?

Sets

7. Set basics:
   (a) What is a *set*? When are two sets considered to be equal?
   (b) What does the notation "$a \in A$" mean, including the context we get from this statement?
   (c) Briefly outline how a set can be presented either *explicitly* or *implictly*, and illustrate with a few examples.
   (d) Name and briefly describe the sets $\varnothing$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.
   (e) If S is a set, what does $|S|$ represent?
   (f) What does the notation $A \subset B$ mean, including the context we get from this statement?
   What word do we use for the relationship between two such sets A and B?

8. Briefly explain how the following sets relate to one another:
   (a) $\{\, 2a \ :\ a \in \mathbb{Z} \,\}$ (b) $\{\, b \in \mathbb{R} \ :\ \frac{b}{2} \in \mathbb{Z} \,\}$ (c) $\{\, \ldots, -4, -2, 0, 2, 4, \ldots \}$ (d) $\{\, 0, 2, -2, 4, -4, \ldots \}$

Fields

9. What does the algebraic concept of *field* naturally generalize? What axioms *define* a field?
   Give a few examples of fields.

10. Simplify the following expressions in the given fields:
    (a) In $\mathbb{Q}$: $-1 + \frac{1}{8}$; $\left(-\frac{3}{7}\right)^{-1}$; $\dfrac{5 + \frac{2}{3}}{\frac{3}{4} - \frac{1}{5}}$
    (b) In $\mathbb{C}$: $(2+i) - (3-i)$; $\frac{1+i}{3+4i}$; $(1+i)^4$
    (*c) In $\mathbb{Z}_2$: 0×1; $1 \times (1+0+1)$; $\left(\frac{1}{1+1+1}\right)^2$
    (*d) In $\mathbb{Z}_5$: -3; $\frac{1}{2}$; 1^4; 2^4; 3^4; 4^4

Linear algebra basics: in principle

1. Within the context of linear algebra, what do call the elements of our field?
   How does this relate to the role they serve in linear algebra?

2. Vector spaces:
   (a) Given a field F of scalars, what is the meaning of a *vector space over F*?
   What axioms *define* a vector space?
   (b) What is the fundamental algebraic operation on vectors in a vector space?
   (c) Give some examples of vector spaces, along with each one's field of scalars.
   (d) What is a *subspace* of a vector space, and what three-point checklist allows us to check whether a collection of vectors from some vector space forms a subspace?

. . . and in practice

3. Briefly explain how vectors could be used to represent of the following situations:
   (a) The chemical reaction of methane burning in oxygen: $CH_4 + 2O_2 \to CO_2 + 2H_2O$
   (b) The risk profiles (% high-, medium-, and low-risk) for various investments
   (c) The combined waveform of three distinct sound waves, amplified by a factor of 3
   (d) A location on a computer display or piece of paper
   (e) The velocity or momentum of a moving object

4. Simplify the following linear combinations in $\mathbb{C}^2$ (over $\mathbb{C}$):
   (a) $2i\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1+i\\-3\end{bmatrix} + i\begin{bmatrix}0\\i\end{bmatrix}$ (b) $0\begin{bmatrix}5i\\6+i\end{bmatrix} + 5\begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}-2\\3i\end{bmatrix}$ (c) $(4-i)\begin{bmatrix}2+i\\1\end{bmatrix} - 3\begin{bmatrix}-1+i\\5-2i\end{bmatrix}$

5. Simplify the following linear combinations in $\mathbb{R}[x]$ (over $\mathbb{R}$):
   (a) $-(1+x+x^2)+2(4+x)$ (b) $1-(1+x)+(x+x^2)$ (c) $2(1-x^2)-4(1-x+x^2)$

*6. Simplify the following linear combinations in $\mathbb{Z}_3[t]$ (over $\mathbb{Z}_3$):
   (a) $-(1-t+t^2)+2(1-t)$ (b) $2(1+t)-(1-2t+2t^2)$ (c) $(t^2+t^4)+1-2(1+t^2)$

7. Determine whether each of the following sets of vectors forms a subspace of $\mathbb{R}^3$ (over $\mathbb{R}$):
   (a) $\left\{ \begin{bmatrix}x\\y\\0\end{bmatrix} : x, y \in \mathbb{R} \right\}$ (b) $\left\{ \begin{bmatrix}x\\y\\1\end{bmatrix} : x, y \in \mathbb{R} \right\}$ (c) $\left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : x, y, z \in \mathbb{R} \text{ and } z = x + y \right\}$
   (d) $\left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : x, y, z \in \mathbb{R} \text{ and } [x = 0 \text{ or } y = 0] \right\}$ (e) $\left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} : x, y, z \in \mathbb{R} \text{ and } x \neq 0 \right\}$

8. Determine whether each of the following sets of vectors forms a subspace of $\mathbb{R}[x]$ (over $\mathbb{R}$):
   (a) $\{ 1 + Ax : A \in \mathbb{R} \}$ (b) $\{ C + Cx : C \in \mathbb{R} \}$ (c) $\{ A + Bx + Cx^2 : A, B, C \in \mathbb{R} \}$

9. Determine whether each of the following sets of vectors forms a subspace of $\mathrm{C}(\mathbb{R})$ (over $\mathbb{R}$):
   (a) $\{ f \in \mathrm{C}(\mathbb{R}) : f(0) = 0 \}$ (b) $\{ f \in \mathrm{C}(\mathbb{R}) : f(0) = 2 \}$ (c) $\{ f \in \mathrm{C}(\mathbb{R}) : f(0) \neq 1 \}$
   (d) $\{ f \in \mathrm{C}(\mathbb{R}) : f(0) = 2f(1) \text{ and } f(3) = 0 \}$

Spans: in principle

Suppose that V is a vector space over a field F, and that $\mathscr{C} = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ is a finite collection of vectors in V.

1. What do we mean by span($\mathscr{C}$), the *span** of $\mathscr{C}$? What does it mean for a vector $\vec{v}$ to be in this span? * noun

2. What does it mean for $\mathscr{C}$ to *span*† V? What does this tell us about the vectors in V? † verb!

3. What property does span($\mathscr{C}$) automatically have? How is this proven?

4. Suppose that $\mathscr{D}$ is another collection of vectors in V.
   (a) How can we prove that the span of $\mathscr{C}$ contains the span of $\mathscr{D}$? Why?
   (b) How can we prove that the spans of $\mathscr{C}$ and $\mathscr{D}$ are the same? Why?

5. Span-preserving manipulations of a collection $\mathscr{C}$:
   What vectors can be *inserted into* or *removed from* a collection $\mathscr{C}$ without changing its span?
   By what can we *replace* any vector of $\mathscr{C}$ without changing its span?

6. Spans in $\mathbb{R}^m$: Suppose that $\mathscr{C}$ is a finite collection of *column vectors* in $\mathbb{R}^m$.
   (a) What does span($\mathscr{C}$) tell us about the linear systems arising from $\mathscr{C}$?
   (b) If $\mathscr{D}$ is another finite collection of column vectors, how can we computationally determine whether or not span($\mathscr{C}$) contains span($\mathscr{D}$)? How can we tell whether the spans of $\mathscr{C}$ and $\mathscr{D}$ are the same?
   (c) How can we determine whether or not $\mathscr{C}$ spans $\mathbb{R}^m$?

. . . and in practice

7. For each statement below:
   (i) Use known properties of the span to argue why the statement must be true *on principle*.
   (ii) Use basic definitions and algebra to show *directly* that the statement is true.
   (a) If $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ spans V, then $\{ \vec{v}_1 - 4\vec{v}_2, \vec{v}_2, \vec{v}_3 \}$ also spans V.
   (b) If $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ and $\{ \vec{v}_1, \vec{v}_2 \}$ both span V, then $\vec{v}_3$ must be a linear combination of $\{ \vec{v}_1, \vec{v}_2 \}$.
   (c) If $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ spans V and $\vec{v}_4$ is any vector in V, then $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \}$ spans V.
   (d) The span of $\{ \vec{v}, \vec{w} \}$ is the same as the span of $\{ 2\vec{v} - \vec{w}, \vec{v} + 3\vec{w} \}$.

8. Suppose that $\mathscr{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ and $\mathscr{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.
   (a) Show that the vector $\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$ is in both span($\mathscr{C}$) and span($\mathscr{D}$).
   (b) Show that span($\mathscr{C}$) contains span($\mathscr{D}$).
   (c) Show that span($\mathscr{D}$) does *not* contain span($\mathscr{C}$), and find a vector of $\mathscr{C}$ that is not in span($\mathscr{D}$).
   (d) Which one of these collections can't possibly span $\mathbb{R}^3$, and why?
   (e) Show that the other collection does span $\mathbb{R}^3$.
   (f) Show that the third vector of $\mathscr{D}$ is a linear combination of the first two.
   (g) Show that the third vector of $\mathscr{C}$ is not a linear combination of the first two.
   (h) Find a condition on a, b, and c that determines whether or not the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in span($\mathscr{D}$).
   [Hint: Reduce the system as usual and check for consistency.]

Linear independence: in principle

Suppose that V is a vector space over a field F, and that $\mathscr{C} = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$ is a finite collection of vectors in V.

1. What is a *linear relation* on $\mathscr{C}$? When is such a linear relation called *trivial*?

2. What does it mean for $\mathscr{C}$ to be *linearly independent*? What does this tell us about the vectors in V, and why?

3. What, then, does it mean for $\mathscr{C}$ to be linearly *dependent*?

4. Manipulation of a linearly independent collection $\mathscr{C}$:
   What vectors can be *inserted into* or *removed from* $\mathscr{C}$ while preserving its linear independence?
   By what can we *replace* any vector of $\mathscr{C}$ while preserving its linear independence?

5. Linear independence and linear systems: suppose that $\mathscr{C}$ is a finite collection of *column vectors* in $\mathbb{R}^m$.
   (a) What does the linear independence of $\mathscr{C}$ tell us about the linear systems arising from $\mathscr{C}$?
   (b) How can we computationally determine whether or not $\mathscr{C}$ is linearly independent?

. . . and in practice

6. Use basic definitions and algebra to show *directly* that each statement below is true:
   (a) Each collection $\mathscr{C}$ containing the zero vector is linearly dependent.
   (b) If $\{ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3 \}$ is linearly *dependent*, then one of the three vectors must be a linear combination of the other two. [Note well that we can't say *which one*!]

7. For each statement below:
   (i) Use known properties of the span to argue why the statement must be true *on principle*.
   (ii) Use basic definitions and algebra to show *directly* that the statement is true.
   (a) If $\{ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3 \}$ is linearly independent, then $\{ \vec{v}_1 - 4\vec{v}_2,\ \vec{v}_2,\ \vec{v}_3 \}$ is also linearly independent.
   (b) If $\{ \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3 \}$ is linearly independent, then so is $\{ \vec{v}_1,\ \vec{v}_2 \}$.
   (c) If $\{ \vec{v},\ \vec{w} \}$ is linearly independent, then so is $\{ 2\vec{v} - \vec{w},\ \vec{v} + 3\vec{w} \}$.

8. Show that the collection $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ is linearly independent.

9. Show that $\left\{ \begin{bmatrix} 4 \\ -12 \\ -8 \end{bmatrix}, \begin{bmatrix} -4 \\ 13 \\ 7 \end{bmatrix}, \begin{bmatrix} -8 \\ 27 \\ 13 \end{bmatrix} \right\}$ is linearly dependent, and find a nontrivial linear relation on it.

10. Determine the value c for which the collection $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ c \end{bmatrix} \right\}$ is linearly *dependent*.
    For this value of c, find a nontrivial linear relation on the resulting collection.

11. Find a condition on a, b, and c that determines whether the collection $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$ is linearly independent or linearly dependent.

Interplay between the span and linear independence: in principle

1. Suppose that $\mathscr{C} = \{\, \vec{v}_1,\, \vec{v}_2,\, \ldots, \vec{v}_n \,\}$ is a collection of vectors in some vector space V; prove that:
   $\mathscr{C}$ is linearly *dependent* if, and only if, one of its vectors lives in the span of the rest.

2. Suppose that $\mathscr{V}$ and $\mathscr{W}$ are finite collections of vectors in a vector space V.
   If $\mathscr{V}$ is linearly independent in V and $\mathscr{W}$ spans V, what relationship must exist between $|\mathscr{V}|$ and $|\mathscr{W}|$?
   How is this proven?

3. Suppose that $\mathscr{C}$ is a collection of vectors in a vector space V.
   (a) If $\mathscr{C}$ spans V, is there any reason to believe that $\mathscr{C}$ is linearly independent?
   (b) Conversely, if $\mathscr{C}$ is linearly independent, is there any reason to believe that $\mathscr{C}$ spans V?
   (c) What do we call a collection $\mathscr{C}$ that both spans V and is linearly independent?

4. If $\mathscr{C}$ is a collection of column vectors in $\mathbb{R}^m$, describe how we can computationally find a basis for the span of $\mathscr{C}$.
   What should we be careful about when writing the basis?
   Briefly explain how we know that this subcollection does the job of a basis.

. . . and in practice

5. Show directly that if $\{\, \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3 \,\}$ spans V and $\vec{v}_4$ is any vector in V, then $\{\, \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \vec{v}_4 \,\}$ is linearly dependent.

6. Show directly that if $\{\, \vec{v}_1,\ \vec{v}_2,\ \vec{v}_3 \,\}$ is linearly independent in V, then $\{\, \vec{v}_1,\ \vec{v}_2 \,\}$ can't possibly span V.

7. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are collections of vectors in a vector space V, and that $\mathscr{C} \subset \mathscr{D}$.
   (a) If $\mathscr{C}$ spans V, what can be concluded about $\mathscr{D}$?
   (b) If $\mathscr{C}$ is linearly independent, what can be concluded about $\mathscr{D}$?
   (c) If $\mathscr{D}$ spans V, what can be concluded about $\mathscr{C}$?
   (d) If $\mathscr{D}$ is linearly independent, what can be concluded about $\mathscr{C}$?

8. Suppose that we have three finite collections of vectors in a vector space V:
   - $\mathscr{C}_{\mathrm{s}}$, which spans V;
   - $\mathscr{C}_{\mathrm{li}}$, which is linearly independent in V; and
   - $\mathscr{C}_{\mathrm{b}}$, which is a basis for V.

   Determine all relationships that must exists among the values $|\mathscr{C}_{\mathrm{s}}|$, $|\mathscr{C}_{\mathrm{li}}|$, and $|\mathscr{C}_{\mathrm{b}}|$.

9. Find a basis for the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 2 \\ -6 \end{bmatrix} \right\}$.

10. Find a basis for the span of $\left\{ \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$.

11. Find a basis for the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} \right\}$.

Basis and dimension: in principle

1. Suppose that $\mathscr{B}$ is a finite collection of vectors in a vector space V.
   What properties must $\mathscr{B}$ have in order to be a *basis* for V? What does this tell us about the vectors of V?

2. Suppose that $\mathscr{B}$ and $\mathscr{B}'$ are finite bases for a vector space V.
   What can be concluded about $|\mathscr{B}|$ and $|\mathscr{B}'|$? Why?
   How does this allow us to define the *dimension* of V as a quantity *intrinsic* to V?

3. Suppose that V is a [finite-dimensional] vector space.
   (a) Describe how, in principle, any finite spanning set $\mathscr{C}$ for V can be reduced to a basis for V.
   (b) Describe how, in principle, any linearly independent set $\mathscr{C}$ in V be extended to a basis for V.
   (c) How do we know that V *has* a basis?

4. Suppose that V is an n-dimensional vector space.
   (a) What is true of any spanning set $\mathscr{C}$ for V with $|\mathscr{C}| = n$, and why?
   (b) What is true of any linearly independent set $\mathscr{D}$ in V with $|\mathscr{D}| = n$, and why?

5. Suppose that $\mathscr{B}$ is a basis for $\mathbb{R}^m$ (note that $\mathscr{B}$ consists of column vectors).
   What does the fact that $\mathscr{B}$ is a basis tell us about the linear systems arising from $\mathscr{B}$?

6. Describe how we can computationally extend any linearly independent collection in $\mathbb{R}^m$ to a basis for $\mathbb{R}^m$.

. . . and in practice

7. Suppose that V is a 7-dimensional vector space.
   (a) If $\mathscr{C}$ spans V, what can be concluded about $|\mathscr{C}|$?
   (b) If $\mathscr{D}$ is linearly independent in V, what can be concluded about $|\mathscr{D}|$?

8. Suppose that $\mathscr{C}$ spans V and has $|\mathscr{C}| = 5$, and that $\mathscr{D}$ is linearly independent in V and has $|\mathscr{D}| = 3$.
   (a) What are the possible dimensions of V?
   (b) For each possible dimension of V, determine what can be concluded about $\mathscr{C}$ and $\mathscr{D}$ in that case.

*9. Suppose that V is a 3-dimensional vector space spanned by $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$.
   If $\{\vec{a}, \vec{c}\}$ is linearly independent in V and $\{\vec{a}, \vec{b}, \vec{c}\}$ is linearly dependent in V, show that $\vec{b} \in \operatorname{span}\{\vec{a}, \vec{c}\}$, then use this information to find a basis for V.

10. Suppose that W is a subspace of a finite-dimensional vector space V.
    What can be said about $\dim W$ vs. $\dim V$? Why?

11. What is the dimension of the vector space $\{\vec{0}\}$, i.e., the vector space whose only vector is the zero vector?

12. Extend the collection $\left\{ \begin{bmatrix} -1 \\ -5 \\ 12 \\ 11 \end{bmatrix}, \begin{bmatrix} -3 \\ -16 \\ 38 \\ 36 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 7 \\ 7 \end{bmatrix} \right\}$ to a basis for $\mathbb{R}^4$.

13. *Try to extend* the collection $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ to a basis for $\mathbb{R}^4$.

    Why does this fail?

Bases and coordinates: in principle

1. The basis as a function: Suppose that $\mathscr{B}$ is an [ordered] basis for an n-dimensional vector space V.
   (a) If $\vec{x} \in \mathbb{R}^n$, how do we define the vector $[\mathscr{B}]\vec{x} \in V$?
   (b) Briefly explain how the fact that $\mathscr{B}$ is a basis tells us that this function $[\mathscr{B}]$ pairs each column vector of $\mathbb{R}^n$ with a particular vector of V, and vice-versa.
   (c) If $\vec{v} \in V$, what does the column vector $[\mathscr{B}]^{-1}\vec{v} \in \mathbb{R}^n$ tell us? What term do we use for this? What does each of its entries give us?

2. What should we be *extremely* careful about when dealing with coordinates and coordinate vectors?

3. Coordinates in $\mathbb{R}^n$: Suppose that $\mathscr{B}$ is an ordered basis for $\mathbb{R}^n$.
   (a) If $\vec{x} \in \mathbb{R}^n$ is a coordinate vector, what simple operation allows us to compute $[\mathscr{B}]\vec{x}$?
   (b) If $\vec{v} \in \mathbb{R}^n$, how can we compute its $\mathscr{B}$-coordinates $[\mathscr{B}]^{-1}\vec{v}$?

4. What are the elements of the *standard basis* for $\mathbb{R}^n$, and how do we denote them?
   What is the relationship between a column vector $\vec{v} \in \mathbb{R}^n$ and its standard coordinates?

5. What is meant by $P_n(x)$? Give a basis for $P_n(x)$; what is the dimension of this vector space?

... and in practice

6. If $\mathscr{B} = \left(\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)$, compute the following:
   (a) $[\mathscr{B}] \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ (b) $[\mathscr{B}] \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ (c) $[\mathscr{B}] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $[\mathscr{B}] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
   (e) $[\mathscr{B}]^{-1} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ (f) $[\mathscr{B}]^{-1} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ (g) $[\mathscr{B}]^{-1} \begin{bmatrix} 7 \\ 9 \\ 14 \end{bmatrix}$ (h) $[\mathscr{B}]^{-1} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

7. Suppose that $\mathscr{B}$ is an ordered basis for an n-dimensional vector space V.
   For any $\vec{v} \in V$, what is $[\mathscr{B}][\mathscr{B}]^{-1}\vec{v}$? For any $\vec{x} \in \mathbb{R}^n$, what is $[\mathscr{B}]^{-1}[\mathscr{B}]\vec{x}$?

8. Find the coordinates of $\vec{v} = \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix} \in \mathbb{R}^3$ with respect to the following bases:
   (a) $\mathscr{B}' = (\vec{e}_2, \vec{e}_1, \vec{e}_3)$ (b) $\mathscr{B}'' = (6\vec{e}_1, \vec{e}_2, \vec{e}_3)$ (c) $\mathscr{B}''' = (\vec{e}_1 + 4\vec{e}_2, \vec{e}_2, \vec{e}_3)$

9. Suppose that $\mathscr{B} = (\,\vec{a},\, \vec{b},\, \vec{c}\,)$ is a basis for a vector space V, and that $\vec{v} \in V$ has $[\mathscr{B}]^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$.
   Find the coordinates of $\vec{v}$ with respect to the following new bases:
   (a) $(\,\vec{b},\, \vec{c},\, \vec{a}\,)$ (b) $(\,2\vec{a},\, -\vec{b},\, \vec{c}\,)$ (c) $(\,\vec{a} + 2\vec{b} - 3\vec{c},\, \vec{b},\, \vec{c}\,)$

10. Consider the bases $\mathscr{B} = \left(1 - t^2,\ t + t^2,\ 2t\right)$ and $\mathscr{B}' = \left(2 - t + t^2,\ 2 + t^2,\ 1 + t + t^2\right)$ for $P_2(t)$.
    (a) If $f(t) \in P_2(t)$ has $\mathscr{B}$-coordinates $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $f(t)$.
    (b) Use your answer above to find the $\mathscr{B}'$-coordinates for $f(t)$.

Vector space isomorphism and the FTVS: in principle

1. Isomorphisms:
   (a) What is an *isomorphism* between two vector spaces V and W?
   (b) Draw a *commutative diagram* illustrating the relationship between an isomorphism and linear combinations.
   (c) If $\phi : V \to W$ is an isomorphism, what must $\phi(\vec{0}_V)$ equal? Why?

2. What does it mean to say that two vector spaces V and W are *isomorphic*?
   What symbol do we use to indicate that V and W are isomorphic?
   Informally, what does this mean about V and W?

3. Explain the following statement (the Fundamental Theorem of [finite-dimensional] Vector Spaces):
   "A basis for an n-dimensional vector space V gives us an isomorphism between $\mathbb{R}^n$ and V."

. . . and in practice

4. Prove the following: [You may assume without proof the bijectivity condition in parts (a) and (b)—just show that the function preserves linear combinations.]
   (a) If $\phi : V \to W$ is an isomorphism, then $\phi^{-1} : W \to V$ is also an isomorphism.
   (b) If $\phi : V \to W$ and $\psi : W \to U$ are isomorphisms, then $\psi \circ \phi : V \to U$ is an isomorphism.
   (c) If V and W are two n-dimensional vector spaces, how do we know that $V \cong W$?
   [Hint: Start with the fact that both are isomorphic to $\mathbb{R}^n$, then use parts (a) and (b)]

5. Suppose that V and W are vector spaces and $\phi : V \to W$ is an isomorphism. Prove the following:
   (a) If $\{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_k \,\}$ spans V, then $\{\, \phi(\vec{v}_1),\, \phi(\vec{v}_2),\, \ldots,\, \phi(\vec{v}_k) \,\}$ spans W.
   (b) If $\{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_k \,\}$ is linearly independent in V, then $\{\, \phi(\vec{v}_1),\, \phi(\vec{v}_2),\, \ldots,\, \phi(\vec{v}_k) \,\}$ is linearly independent in W.
   (c) If $\{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_k \,\}$ is a basis for V, then $\{\, \phi(\vec{v}_1),\, \phi(\vec{v}_2),\, \ldots,\, \phi(\vec{v}_k) \,\}$ is a basis for W.

   Given what you've proven above, what can you conclude about the dimensions of isomorphic vector spaces?

6. In light of the previous two problems, what is the unique *fundamental* property intrinsic to a finite-dimensional vector space, considered simply as abstract vector space (i.e., "up to isomorphism")? Briefly explain.

7. Given the basis $\mathscr{B} = \big(1 - x,\ x + 2x^2,\ 3 + x^2\big)$ for $P_2(x)$, verify directly that if $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then $[\mathscr{B}](2\vec{x}_1 - 3\vec{x}_2) = 2[\mathscr{B}]\vec{x}_1 - 3[\mathscr{B}]\vec{x}_2$.

8. Given the basis $\mathscr{B} = \big(1,\ x,\ x^2\big)$ for $P_2(x)$, verify directly that if $\vec{v}_1 = 3x^2 + x + 2$ and $\vec{v}_2 = x - x^2 + 1$, then $[\mathscr{B}]^{-1}(-4\vec{v}_1 + 2\vec{v}_2) = -4[\mathscr{B}]^{-1}\vec{v}_1 + 2[\mathscr{B}]^{-1}\vec{v}_2$.

9. Suppose that $\mathscr{B}$ is a basis for a 4-dimensional vector space V, and that the $\mathscr{B}$-coordinates of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_5 \in V$ are:
   $$[\mathscr{B}]^{-1}\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 6 \end{bmatrix}; \quad [\mathscr{B}]^{-1}\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 2 \end{bmatrix}; \quad [\mathscr{B}]^{-1}\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 9 \\ 18 \end{bmatrix}; \quad [\mathscr{B}]^{-1}\vec{v}_4 = \begin{bmatrix} -2 \\ 0 \\ 2 \\ -8 \end{bmatrix}; \quad [\mathscr{B}]^{-1}\vec{v}_5 = \begin{bmatrix} 4 \\ 2 \\ 6 \\ 14 \end{bmatrix}.$$
   (a) Is the collection $\{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_5 \,\}$ linearly independent? If not, find a nontrivial linear relation on it.
   (b) Does the collection $\{\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_5 \,\}$ span V?
   (c) Reduce the collection $\{\, \vec{v}_1,\, \vec{v}_2,\, \vec{v}_3,\, \vec{v}_4,\, \vec{v}_5 \,\}$ to a basis for V.

   Do you know what the vectors of V represent? Did this affect your ability to answer the questions? Why?

Computation in abstract vector spaces

1. Via the FTVS, how can we approach computational questions in [finite-dimensional] abstract vector spaces? What is the thing we need in order to do so? What should we be careful to do when finished?

. . . and in practice

In each problem below, find an appropriate basis for the relevant vector space, then use this to reëxpress the problem in terms of linear combinations of column vectors and solve the problem.

2. Consider the polynomials $1 + x$, $x + x^2$, and $1 + x^2$.
   (a) Do these polynomials span $P_2(x)$? Are they linearly independent? Do they form a basis for $P_2(x)$?
   (b) Write x^2 as a linear combination of these polynomials.
   (c) Can we write *every* polynomial in $P_2(x)$ as a linear combination of these polynomials? In how many ways? Why?

3. Consider the polynomials $1 - x$, $x - x^2$, and $1 - x^2$.
   (a) Show that $1 - 2x + x^2$ can be written as a linear combination of these polynomials.
   (b) Show that $1 - x + x^2$ *can't* be written as a linear combination of these polynomials.
   (c) Find a basis for the span of these polynomials; what is the dimension of their span?
   Interpret your answers to parts (a) and (b) in terms of this span.
   (d) Find the condition on a, b, and c that determines whether a polynomial $a+bx+cx^2$ can be written as a linear combination of these polynomials.

4. Consider the collection $\mathscr{C} = \{ 1 + t + t^3,\ t + t^2 - t^3,\ 1 - t^2 \}$ of polynomials in $P_3(t)$.
   (a) Show that $\mathscr{C}$ is linearly independent.
   (b) Extend $\mathscr{C}$ to a basis for $P_3(t)$.

5. Consider the collection $\mathscr{C} = \{ 1 - 2z + 3z^2,\ 13z^2 + 6z - 1,\ 23z^2 + 14z - 3,\ 2 - 12z - 23z^2 \}$ in $P_2(z)$.
   (a) Show that $\mathscr{C}$ is linearly dependent, and find a nontrivial linear relation on $\mathscr{C}$.
   (b) Reduce $\mathscr{C}$ to a basis for $P_2(z)$.

6. Suppose that the following three [equally-priced] mutual funds are available, whose risk breakdowns are as follows:
   - Fund A: 20% high-risk, 50% medium-risk, and 30% low-risk
   - Fund B: 20% high-risk, 10% medium-risk, and 70% low-risk
   - Fund C: 50% high-risk, 40% medium-risk, and 10% low-risk

   If you want to invest so that your risk is spread evenly between low-, medium-, and high-risk investments, in what proportions should you buy these three funds?

7. Balance the equations for the following chemical reactions (be sure that your final answers are positive integers!):
   (a) Aluminum hydroxide [$Al(OH)_3$] and sulfuric acid [H_2SO_4] reacting to form aluminum sulfate [$Al_2(SO_4)_3$] and water [H_2O].
   (b) Ethanol [C_2H_5OH] burning in oxygen [O_2] to form carbon dioxide [CO_2] and water [H_2O].
   (*c) Potassium chlorate [$KClO_3$] and hydrogen chloride [HCl] reacting to form potassium chloride [KCl], water [H_2O], chlorine [Cl_2], and chlorine dioxide [ClO_2].

   What basis could you take in order to allow *any* chemical reaction to be expressed?
   What is the dimension of the relevant vector space?

Function basics

1. Suppose that X and Y are sets.
   (a) What do we mean by a *function* f from X to Y? How do we denote this?
   What are the *domain* and *codomain* of f?
   (b) How is this idea turned into a precise logical definition?

2. If $f : X \to Y$ is a function, what do we mean by its *range*?
   How do we formally define the range of f?
   Briefly explain the relationship between a function's codomain and its range.

3. Suppose that $f : X \to Y$ is a function. Define and briefly explain what it means for f to be:
   (a) *injective* (b) *surjective* (c) *bijective*

4. Given functions $f : X \to Y$ and $g : Y \to Z$, how do we define their *composition* $g \circ f$?

5. Given two functions $f_1, f_2 : X \to Y$, what does it mean to say that $f_1 = f_2$?

6. If X is a set, how do we define the *identity function* on X? How do we denote it?
   Prove the following properties of identity functions:
   (a) id_X is bijective.
   (b) If $f : X \to Y$, then $f \circ \mathrm{id}_X = f$.
   (c) If $f : X \to Y$, then $\mathrm{id}_Y \circ f = f$.

7. What is an *inverse* for f? How do we denote it? What does it mean for f to be *invertible*?

8. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. Prove the following:
   (a) ...about injectivity:
      (i) If f is injective and g is injective, then $g \circ f$ is injective.
      (ii) If $g \circ f$ is injective, then f is injective. [Why needn't g be injective?]
   (b) ...about surjectivity:
      (i) If f is surjective and g is surjective, then $g \circ f$ is surjective.
      (ii) If $g \circ f$ is surjective, then g is surjective. [Why needn't f be surjective?]
   (c) ...about bijectivity: [Don't work too hard—use what you've proven in parts (a) and (b)!]
      (i) If f is bijective and g is bijective, then $g \circ f$ is bijective.
      (ii) If $g \circ f$ is bijective, then f is injective and g is surjective.
   (d) ...about inverses:
      (i) If $f : X \to Y$ is invertible, then f and f^{-1} are both bijective.
      [Hint: id_X and id_Y are bijective; use part (c) twice]
      (ii) If f and g are invertible, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
      [Hint: Composing with an identity function has no effect.]

Linear transformation basics: in principle

Suppose that V and W are vector spaces.

1. What is a *linear transformation* from V to W? What are its *domain* and *codomain*?
   By what three-point checklist can we check whether a function from V to W is a linear transformation?

2. The image of a LT: Suppose that $L : V \to W$ is a linear transformation.
   (a) What is the *image* of L? How do we denote it? What does it tell us about the function L?
   (b) Prove that $\operatorname{im} L$ is a subspace of W.
   (c) If $\operatorname{im} L = W$, what does this tell us about the function L?

3. The kernel of a LT: Suppose that $L : V \to W$ is a linear transformation.
   (a) What is the *kernel* of L? How do we denote it? What does it tell us about the function L?
   (b) Prove that $\ker L$ is a subspace of V.
   (c) If $\ker L = \{\vec{0}\}$, what does this tell us about the function L?

4. Composition of LT's: Suppose that $L : V \to W$ and $K : W \to U$ are linear transformations.
   (a) How do we define the *composition* $K \circ L$? What are its domain and codomain?
   (b) Prove that $K \circ L$ is a linear transformation.

. . . and in practice

5. Suppose that $L : V \to W$ is a linear transformation.
   (a) Show that $L(\vec{x}) = L(\vec{y})$ if, and only if, $\vec{x} - \vec{y} \in \ker L$.
   (b) Show that if $L(\vec{x}_0) = \vec{a}$, then: $L(\vec{x}) = \vec{a}$ if, and only if, there is some $\vec{v} \in \ker L$ with $\vec{x} = \vec{x}_0 + \vec{v}$.
   * In words: if you know just one solution x_0 along with the kernel, then you know *all* solutions!

6. Suppose that $L : V \to W$ and $K : W \to U$ are linear transformations.
   Show that $\vec{x} \in \ker(K \circ L)$ if, and only if, $L(\vec{x}) \in \ker K$.

7. Determine whether or not each of the following functions is a linear transformation, and justify your answers:
   (a) $L : \mathbb{R}^3 \to \mathbb{R}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto xy + z$
   (b) $L : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3x - y + z \\ z + y - 2x \end{bmatrix}$
   (c) $L : \mathrm{C}(\mathbb{R}) \to \mathbb{R}^2$, $f \mapsto \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}$
   (d) $L : \mathrm{C}(\mathbb{R}) \to \mathbb{R}$, $f \mapsto f(0) + 1$
   (e) $L : \mathrm{C}(\mathbb{R}) \to \mathbb{R}$, $f \mapsto 2f(0) - f(1)$
   (f) $L : \mathbb{R}^2 \to \mathbb{R}[x]$, $\begin{bmatrix} A \\ B \end{bmatrix} \mapsto A + Bx + (A + B)x^2$

8. Determine the kernels of the linear transformations given in parts (b) and (f) of the previous problem.

9. Suppose that $L : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x - y + z \end{bmatrix}$, and that $K : \mathbb{R}^2 \to \mathbb{R}$ is given by $\begin{bmatrix} u \\ v \end{bmatrix} \mapsto u - 3v$.
   (a) What are the domain and codomain of $K \circ L$?
   (b) Evaluate and simplify the expression $(K \circ L) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
   (c) Determine the kernel of $K \circ L$.

Linear transformations: from collections

Suppose that $\mathscr{C} = (\,\vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_n\,)$ is an ordered collection of vectors in some vector space V.
The following problems concern the function $[\mathscr{C}]$.

1. $[\mathscr{C}]$ as a linear transformation:
   (a) How do we define $[\mathscr{C}]$? What are its domain and codomain?
   (b) Show that $[\mathscr{C}]$ is a linear transformation.

2. $\mathscr{C}$ vis-à-vis the kernel and image of $[\mathscr{C}]$:
   (a) What does $\operatorname{im}[\mathscr{C}]$ give us, in terms of the collection $\mathscr{C}$? Why?
   (b) What does $\ker[\mathscr{C}]$ give us, in terms of the collection $\mathscr{C}$? Why?

3. $[\mathscr{C}]$ and properties of the collection $\mathscr{C}$:
   (a) If $\mathscr{C}$ is linearly independent in V, what does this tell us about this function $[\mathscr{C}]$? Why?
   (b) If $\mathscr{C}$ is spans V, what does this tell us about the function $[\mathscr{C}]$? Why?
   (c) If $\mathscr{C}$ is a basis for V, what does this tell us about the function $[\mathscr{C}]$?
   What do we call such a function?

. . . and from matrices

Suppose now $\mathscr{C} = (\,\vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_n\,)$ is an ordered collection of vectors in $\mathbb{R}^m$ (so that each $\vec{v}_j$ is a *column vector*), and that we write $[\mathscr{C}]$ as a *matrix* A.

4. Answer the following questions about the matrix A:
   (a) Many rows does A have? How many columns? What are its *dimensions*?
   (b) If $\vec{x} \in \mathbb{R}^n$, what is the meaning of $A\vec{x}$?
   (c) Considered as a linear transformation, what are the domain and codomain of A?

5. Define the *nullspace* $N(A)$—what is another term for $N(A)$, considering A as a linear transformation?
   Of what vector space is $N(A)$ a subspace? How can we find a basis for $N(A)$?

6. Define the *column space* $C(A)$—what is another term for $C(A)$, considering A as a linear transformation?
   Of what vector space is $C(A)$ a subspace? How can we find a basis for $C(A)$?

7. Compute the following:
   (a) $\begin{bmatrix} 2 & 1 \\ -3 & 0 \\ -1 & 5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
   (b) $\begin{bmatrix} 3 & 0 & 1 & 4 \\ -2 & -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$
   (c) $\begin{bmatrix} 4 & 1 & 3 & 0 \\ 2 & -2 & 0 & 5 \\ 1 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

8! What does the value of $A\vec{e}_j$ give us from A?
   Let $A = \begin{bmatrix} 3 & 0 & 1 & 4 \\ -2 & -2 & 1 & 5 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$; write $\vec{x}$ as a linear combination of $\vec{e}_j$'s, and use this to compute $A\vec{x}$ (compare with problem 7(b)).

9. Explain what each of the following says about the collection consisting of the columns of an $m \times n$ matrix A:
   (a) $A\vec{x} = \vec{0}$ only for $\vec{x} = \vec{0}$.
   (b) $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$.
   (c) $A\vec{x} = \vec{b}$ has a solution.
   (d) $A\vec{x} = \vec{b}$ has no solutions.
   (e) For each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a *unique* solution.

Rank and nullity: in principle

Suppose that V and W are finite-dimensional vector spaces and $L : V \to W$ is a linear transformation.

1. Prove that if $\{\, \vec{a}_1, \ldots, \vec{a}_m \,\}$ spans V, then $\{\, L\vec{a}_1, \ldots, L\vec{a}_m \,\}$ spans $\underline{\operatorname{im} L}$.

2. How do we define the *rank* and *nullity* of L? Within what range do these values lie? Why?

3. Prove the Rank+Nullity Theorem, via the following steps:
   (a) Take a basis $\{\, \vec{k}_1, \ldots, \vec{k}_n \,\}$ for $\ker L$—what do we call the number n?
   (b) Append vectors $\vec{v}_1, \ldots, \vec{v}_r \in V$ to extend this collection to a basis $\{\, \vec{k}_1, \ldots, \vec{k}_n; \vec{v}_1, \ldots, \vec{v}_r \,\}$ for V, and conclude that $n + r = \dim V$.
   (c) Show that the collection $\{\, L\vec{k}_1, \ldots, L\vec{k}_n; L\vec{v}_1, \ldots, L\vec{v}_r \,\}$ spans $\operatorname{im} L$,
   and conclude that $\{\, L\vec{v}_1, \ldots, L\vec{v}_r \,\}$ spans $\operatorname{im} L$ as well.
   (d) Show that the collection $\{\, L\vec{v}_1, \ldots, L\vec{v}_r \,\}$ is linearly independent,
   and conclude that $\{\, L\vec{v}_1, \ldots, L\vec{v}_r \,\}$ is a basis for $\operatorname{im} L$—what do we thus call the number r?
   (e) Complete the statement of the Rank+Nullity Theorem:
   If V is a finite-dimensional vector space and $L : V \to W$ is a linear transformation, then ...

4. In the case of an $m \times n$ matrix A, briefly explain how our row-reduction methods demonstrate the rank+nullity theorem for the linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$.

... and in practice

In problems 5 and 6, suppose that $L : V \to W$ is a linear transformation, where V is n-dimensional and W is m-dimensional.

5. What can be concluded about L if:
   (a) ... the nullity of L is 0?
   (b) ... the nullity of L is n?
   (c) ... the rank of L is 0?
   (d) ... the rank of L is m?

6. In light of your answers above, if $n < m$, can L be surjective?
   If $n > m$, can L be injective? Why or why not?

7. For each of the following matrices A:
   (i) Determine the domain and codomain of the linear transformation given by A.
   (ii) Determine bases for $N(A)$ and $C(A)$.
   (iii) Find the rank and nullity of the linear transformation A,
   and check the Rank+Nullity Theorem's assertion about their values.
   (iv) Determine whether A is injective, surjective, bijective, or neither.

(a) $A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & -1 & -4 \\ 1 & 0 & 7 \\ -2 & 1 & -7 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -2 & -4 & -10 \\ 3 & -5 & -10 & -27 \\ -2 & 6 & 13 & 29 \end{bmatrix}$ (c) $A = \begin{bmatrix} -1 & -2 & -1 & -3 & 8 \\ 1 & 3 & -2 & 2 & -2 \\ 0 & 1 & -5 & -3 & 12 \\ -1 & -3 & 0 & -4 & 8 \end{bmatrix}$

Linear transformations of $\mathbb{R}^n$, and the FTLT: in principle

1. Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.
   If we define an $m \times n$ matrix A by setting its j^{th} column to the value of $L\vec{e}_j$, show that $\forall\, \vec{x} \in \mathbb{R}^n,\ A\vec{x} = L\vec{x}$.
   What does this allow us to conclude about linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$?

2. Suppose that:
   - V is an n-dimensional vector space, with [ordered] basis $\mathscr{V}$;
   - W is an m-dimensional vector space, with [ordered] basis $\mathscr{W}$; and
   - $L : V \to W$ is a linear transformation.

   (a) How can we use these ingredients to define the linear transformation ${}_{\mathscr{W}}[L]_{\mathscr{V}} : \mathbb{R}^n \to \mathbb{R}^m$?
   What do we know about such a linear transformation?

   (b) State the Fundamental Theorem of Linear Transformations [of finite-dimensional vector spaces].

   (c) Draw a commutative diagram illustrating the workings of the FTLT.

   (d) What does the j^{th} column of the the matrix ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ represent?
   How does this allow us to compute ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ for any given L, $\mathscr{V}$, and $\mathscr{W}$?

   (e) Is ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ intrinsic to L? Why or why not?

. . . and in practice

3. Find the matrix representations of the following linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$:

   (a) $L : \mathbb{R}^3 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 3x - y + z \\ z + y - 2x \end{bmatrix}$

   (b) $L : \mathbb{R}^4 \to \mathbb{R}^3,\ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \mapsto \begin{bmatrix} z + x - 2y \\ 10y + 2x - z + 3w \\ w + 7x - y \end{bmatrix}$

   (c) $L : \mathbb{R}^2 \to \mathbb{R}^3,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x + y \\ y - x \\ 3x \end{bmatrix}$

   (d) $L : \mathbb{R}^3 \to \mathbb{R}^4,\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x + y + z \\ x + 2y + 3z \\ x + 4y + 9z \\ z - y \end{bmatrix}$

4. This problem concerns the linear transformation of *differentiation.*
   For each pair V, W of vector spaces below, find the matrix ${}_{\mathscr{W}}[D]_{\mathscr{V}}$ of the differentiation operation D, with respect to the given bases for V and W.

   (a) $V = P_3(x)$, $W = P_2(x)$ [via the usual bases]

   (b) $V = P_3(x)$, $W = P_2(x)$ [via bases $\mathscr{V} = (\,x,\ x^2,\ x^3,\ 1\,)$ and $\mathscr{W} = (\,1,\ 2x,\ 3x^2\,)$]

   (c) $V = W = \operatorname{span}(\,\sin x,\ \cos x,\ \sin 2x,\ \cos 2x\,) \subset \mathrm{C}(\mathbb{R})$. [The given collection is a basis.]

   (d) $V = W = \operatorname{span}(\,e^x,\ xe^x,\ e^{-x},\ xe^{-x}\,) \subset \mathrm{C}(\mathbb{R})$. [The given collection is a basis.]

5. Rework parts (c) and (d) of the previous problem, now for the linear transformation $f(x) \mapsto f(x) + f''(x)$
   [Hint: Here, f'' denotes the second derivative of f].

6. Find the matrix representation (with respect to the usual bases) of the linear transformation $L : P_3(x) \to P_3(x)$, $f(x) \mapsto xf'(x) - 2f(x)$.

Solving computational problems for linear transformations: in principle

1. Via the FTLT, how can we approach computational questions involving abstract linear transformations?
   What two things do we need in order to start?
   What should we be careful to do when finished?

. . . and in practice

In part (a) of each problem below:

- Find the matrix representation of the linear transformation with respect to appropriate bases.
- Determine bases for the image and kernel of the linear transformation.
- Determine the linear transformation's rank and nullity.
- Determine whether the linear transformation is injective, surjective, and/or bijective.

2. Let $T : P_3(x) \to P_3(x)$ be defined by $f(x) \mapsto xf''(x) + f'(x)$.
   (a) Analyze T as above.
   (b) Find all polynomials $f(x) \in P_3(x)$ for which $T(f(x)) = 1 + x + x^2$; how does your solution relate to $\ker T$?

3. Let $V = \operatorname{span}(\, e^x,\ xe^x,\ x^2e^x,\ x^3e^x \,) \subset \mathrm{C}(\mathbb{R})$,
   and define $L : V \to V$ by $f(x) \mapsto f''(x) - 2f'(x) + f(x)$.
   (a) Analyze L as above.
   (b) Find all $f(x) \in V$ for which $L(f(x)) = 3e^x - 2xe^x$; how does your solution relate to $\ker L$?

4. Suppose $V = \operatorname{span}(\, \sin x,\ \sin 2x,\ \sin 3x \,) \subset \mathrm{C}(\mathbb{R})$,
   and define $T : V \to V$ by $f(x) \mapsto 4f(x) + f''(x)$.
   (a) Analyze T as above.
   (b) Find all $f(x) \in V$ for which $T(f(x)) = \sin x + \sin 3x$; how does your solution relate to $\ker T$?

5. Define $T : P_2(x) \to \mathbb{R}^3$ by $f(x) \mapsto \begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$.
   (a) Analyze T as above.
   (b) Find a polynomial $f(x) \in P_2(x)$ such that $T(f(x)) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.
   (c) Find a quadratic polynomial whose graph passes through the points $(0, 1)$, $(1, 3)$, and $(2, 2)$.
   [Hint: Remember that the points on the graph have the form $(x, f(x))$, and see part (b)!]
   How many such quadratic polynomials are there?
   (d) Find the general formula for the polynomial of degree ≤ 2 whose graph passes through the points $(0, a)$, $(1, b)$, and $(2, c)$. [Hint: Work exactly as above, but now with a, b, and c]
   (e) Find the polynomial of degree ≤ 2 whose graph:
   (i) . . . passes through the points $(0, 2)$, $(1, 1)$, and $(2, 0)$.
   (ii) . . . passes through the points $(0, 10)$, $(1, 4)$, and $(2, -1)$.

Composition and inverses: of linear transformations

1. In order for a linear transformation $L : V \to W$ to be invertible, what must be true of V, W, and the rank of L? Why? What do we call such a linear transformation?

2. Suppose that $L : V \to W$ and $K : W \to U$ are linear transformations, where V, W, and U are finite-dimensional vector spaces. Prove the following statements via definitions and the basic results of Problem Set 11:
   (a) $\operatorname{im}(K \circ L) \subset \operatorname{im} K$, and its corollary: $\operatorname{rank}(K \circ L) \leq \operatorname{rank} K$.
   (*b) $\operatorname{rank}(K \circ L) \leq \operatorname{rank} L$ [Hint: Start with a basis for V as in the proof of the Rank+Nullity Theorem.]
   (c) $\ker L \subset \ker(K \circ L)$, and its corollary: $\operatorname{nullity}(K \circ L) \geq \operatorname{nullity} L$.

3. What is our conceptual framework for analyzing rank and nullity of compositions?

4. Suppose that you are given vector spaces V, W, U, where: ▪ $\dim V = 4$ ▪ $\dim W = 3$ ▪ $\dim U = 5$
   and linear transformations L, K, M, N, J as follows:
   - $L : V \to W$, with rank 3
   - $N : V \to V$, with rank 4
   - $K : W \to U$, with rank 3
   - $R : W \to W$, with rank 3
   - $M : U \to V$, with rank 1
   - $J : U \to U$, with rank 4

   (a) Determine the nullity of each of these linear transformations.
   (b) Classify each of these linear transformations as injective, surjective, bijective, or neither.
   (c) What can be determined about the the following compositions (rank, nullity, and properties)?
      (i) $K \circ L$ (ii) $M \circ K$ (iii) $L \circ M$ (iv) $R \circ L \circ N$
      (v) $M \circ J$ (vi) $N \circ N$ (vii) $J \circ K \circ R$ (viii) $J \circ J \circ J$
      (ix) $J \circ J \circ J \circ J \circ J$

. . . and of matrices

5. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix.
   (a) As a linear transformation, what are the domain and codomain of A? What are they for B?
   (b) What is then meant by the matrix AB, and how does our view of matrices as linear transformations dictate the rule for composing matrices?
   (c) How do we compute the j^{th} column of the composition AB?

6. What is the *identity matrix* of size n? How do we denote it? What collection does it correspond to? What linear transformation of $\mathbb{R}^n$ does it give us? What properties does it have?

7. Suppose A is a matrix.
   (a) What is an *inverse* for A, and how do we denote it? What does it mean for A to be *invertible*?
   (b) In order for A to be invertible, what must be true of its size, rank, and nullity?
   (c) How can we use matrix reduction to *compute* the inverse of an invertible matrix A?
   Why does this work? What happens if we try this procedure with a non-invertible matrix?
   (d) A priori, the method above only gives us a *right-inverse* for A.
   Show that it is also a left-inverse for A.

8. If we have an inverse for a matrix A, how can we quickly solve the linear system $A\vec{x} = \vec{b}$?
   Why is this an extremely specialized approach to solving linear systems?

Change of basis: in principle

1. Change of basis: suppose that $L : V \to W$ is a linear transformation.
   (a) If $\mathscr{V}, \mathscr{V}'$ are two bases for an n-dimensional vector space V and $\mathscr{W}, \mathscr{W}'$ are two bases for an m-dimensional vector space W, what is the formula relating the matrices ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ and ${}_{\mathscr{W}'}[L]_{\mathscr{V}'}$?
   Draw a commutative diagram illustrating this relationship.
   (b) What is a change-of-basis matrix? How do we compute one, in general?
   What properties must a change-of-basis matrix have?
   (c) What is the relationship between the matrix for a given change of basis and the matrix for the change in the other direction? Why?
   (d) How can matrix reduction be used to compute case-of-change of basis matrices in $\mathbb{R}^n$?
   (e) Briefly explain the following statement:
   "A change-of-basis matrix is a matrix representation of the identity linear transformation."

2. The equivalence problem:
   (a) What question of equivalence for $m \times n$ matrices does change of basis present us with?
   (b) By choosing particularly "nice" bases for V and W what form can we force ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ into?
   What properties of L dictate exactly what such a matrix for L must look like?
   How can we, then, quickly determine the answer to our equivalence question?

. . . and in practice

3. Given bases $\mathscr{V} = (\,\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\,)$ and $\mathscr{V}' = (\,-\vec{v}_3 + \vec{v}_4, \vec{v}_3 - 3\vec{v}_4, 2\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2\,)$ for a vector space V, and bases $\mathscr{W} = (\,\vec{w}_1, \vec{w}_2, \vec{w}_3\,)$ and $\mathscr{W}' = (\,3\vec{w}_1 + 2\vec{w}_2 - \vec{w}_3, \vec{w}_1 + \vec{w}_2 - \vec{w}_3, -2\vec{w}_1 + 3\vec{w}_2\,)$ for a vector space W, compute the following change-of-basis matrices:
   (a) $[\mathscr{V}]^{-1}[\mathscr{V}']$ (b) $[\mathscr{V}']^{-1}[\mathscr{V}]$ (c) $[\mathscr{W}]^{-1}[\mathscr{W}']$ (d) $[\mathscr{W}']^{-1}[\mathscr{W}]$

4. With $\mathscr{V}, \mathscr{W}, \mathscr{V}', \mathscr{W}'$ as above, suppose that $L : V \to W$ has ${}_{\mathscr{W}}[L]_{\mathscr{V}} = \begin{bmatrix} -2 & -2 & -29 & -11 \\ 3 & 3 & -20 & -8 \\ 0 & 0 & 11 & 5 \end{bmatrix}$.
   (a) Find ${}_{\mathscr{W}'}[L]_{\mathscr{V}'}$.
   (b) In terms of the bases $\mathscr{V}'$ and $\mathscr{W}'$, describe in words, as simply as possible, what the linear transformation L does.
   (c) Briefly explain the utility of change-of-basis as a tool for studying the behavior of linear transformations.

5. Find the matrix for change-of-basis from the standard basis for $\mathbb{R}^4$ to each of the following bases:
   (a) $(\,\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 - \alpha\vec{e}_2\,)$ $[\,\alpha \in \mathbb{R}\,]$
   (b) $(\,\vec{e}_2, \vec{e}_1, \vec{e}_3, \vec{e}_4\,)$
   (c) $(\,\vec{e}_1, \frac{1}{\alpha}\vec{e}_2, \vec{e}_3, \vec{e}_4\,)$ $[\,\alpha \in \mathbb{R}, \alpha \neq 0\,]$

*6. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}$.
   (i) Apply each change-of-basis matrix found in the previous problem to [the codomain side of] the matrix A, and describe in words what effect effects each of these changes of basis has on the matrix.
   (ii) Briefly explain the perspective thus afforded us (on matrix reduction and solving linear systems) by the concept of change of basis.

Endomorphisms: similarity and diagonalization, in principle

1. Endomorphisms: let V be a vector space.
   (a) What is an *endomorphism* of V?
   (b) What can we do with an endomorphism that we can't do with other linear transformations?
   (c) How do we adjust matrix representations and the question of change of basis in the case of an endomorphism? Why?
   (d) Define what it means for two $n \times n$ matrices A and B to be *similar*. How do we denote this?

2. Suppose that L is an endomorphism of V and $\mathscr{V}$ is a basis for V.
   (a) What is $[L^k]_{\mathscr{V}}$? Why?
   (b) If L is an *automorphism* of V (i.e., invertible endomorphism of V, i.e., isomorphism between V and itself), what is $[L^{-1}]_{\mathscr{V}}$? Why?

3. What does it mean for an endomorphism $L : V \to V$ to be *diagonalizable*?
   What is so nice about a diagonalization of L?

4. What does it mean for two endomorphisms A and B to *commute*?
   What do the endomorphisms of $\mathbb{R}^2$ given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ show us?

. . . and in practice

5. Suppose that A, B, C are $n \times n$ matrices. Prove the following properties of similarity:
   (a) $A \sim A$. (b) If $A \sim B$, then $B \sim A$. (c) If $A \sim B$ and $B \sim C$, then $A \sim C$.
   (d) $A \sim I_n$ if, and only if, $A = I_n$. [Properties (a)–(c) are those of an *equivalence relation*]

6. Suppose that an endomorphism $L : V \to V$ is diagonalizable, with $[L]_{\mathscr{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ for some basis $\mathscr{V} = (\,\vec{v}_1, \ldots, \vec{v}_n\,)$ for V.
   (a) What is $L\vec{v}_1$? $L\vec{v}_2$? $L\vec{v}_n$?
   (b) Find formulae for $[L^2]_{\mathscr{V}}$ and, in general, $[L^k]_{\mathscr{V}}$, where $k > 0$.
   (c) What condition on the λ_i's determines whether or not L is invertible?
   If L is invertible, what is $[L^{-1}]_{\mathscr{V}}$?

7. If A and B are two endomorphisms of some vector space V, expand $(A + B)^2$ and $(A + B)^3$.
   How do these expressions simplify if A and B commute?

8. A square matrix A is called *nilpotent* if $\exists\, k > 0$ such that $A^k = 0$ (the zero matrix).
   (a) Prove that if A is nilpotent and $A \sim B$, then B is nilpotent.
   (b) If A is nilpotent and diagonalizable, what can be concluded about A?
   (c) Prove that if A is nilpotent, then $I - A$ is invertible. [Oblique hint: $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$]

9. A square matrix A has *finite order* if $\exists\, k > 0$ such that $A^k = I$ (the identity matrix).
   (a) Prove that A has finite order and $A \sim B$, then B has finite order.
   (b) Prove that if A has finite order, then A is invertible—what is its inverse?
   (c) If A is diagonalizable and has finite order, what can be concluded the diagonal entries that appear in its diagonalization?

Eigenvalues and eigenvectors I: in principle

Suppose that $L : V \to V$ is an endomorphism, where V is an n-dimensional vector space.

1. Define the terms *eigenvalue* and *eigenvector* of L.

2. Briefly explain the relationship between eigenvectors, bases, and diagonalizability of L.

*3. Prove that if $\mathscr{C} = \{\, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \,\}$ consists of eigenvectors of L with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $\mathscr{C}$ is linearly independent.
What can thus be concluded if L has n distinct eigenvalues?

4. Suppose that A is a square matrix.
   (a) Prove that $\vec{x} \neq \vec{0}$ is an eigenvector of A with eigenvalue λ if, and only if, $\vec{x} \in \ker(A - \lambda I)$.
   (b) How can we thus determine the eigenvectors for a given eigenvalue of A?

5. What is an *upper-triangular* matrix?
What can be said about the eigenvalues of an upper-triangular matrix? Why?

. . . and in practice

6. Consider the endomorphism given by $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$.
What must the eigenvalues of A be? Find the associated eigenvectors; why can't A be diagonalized?

7. Suppose that $L : V \to V$ is an endomorphism, and for each $\lambda \in \mathbb{R}$, define $E_\lambda = \{\, \vec{v} \in V : L\vec{v} = \lambda\vec{v} \,\}$.
   (a) Show that E_λ is a *subspace* of V. [E_λ is called λ's *eigenspace* in V.]
   (b) What can be concluded if $\dim E_\lambda > 0$? What if $\dim E_\lambda = 0$?

8. Suppose $L : V \to V$ is an endomorphism, and that 4 is the only eigenvalue of L^2.
What can be concluded about the eigenvalues of L? Why?

9. If a nonzero vector $\vec{v} \in V$ is an eigenvector of $L : V \to V$ with eigenvalue 0, what does this tell us about $\vec{v}$?
Consequently, what can be concluded about the eigenvalues of an *injective* endomorphism $L : V \to V$?

10. Suppose $V = \operatorname{span}(\sin x,\ \sin 2x,\ \sin 3x) \subset \mathrm{C}(\mathbb{R})$, and define $T : V \to V$ by $f(x) \mapsto 4f(x) + f''(x)$.
Use your work from Problem Set 13, #4 to determine the eigenvalues of T and find an eigenbasis for V.

11. Let $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.
   (a) Find the eigenvalues of A, and use these to find an eigenbasis for $\mathbb{R}^4$.
   (b) Find a diagonalization $A = X\Lambda X^{-1}$, and compute the composition to directly verify that it works.

12. Do as in the problem above, but for the matrix $B = \begin{bmatrix} 3 & 1 & 5 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Multilinear forms [optional, except for the definitions and basic facts]

Suppose that V is an n-dimensional vector space, and that $\mathscr{B} = (\,\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\,)$ is a basis for V.

1. What is a *multilinear form* on V?

2. Considering bases and multilinearity, how many free choices do we have when constructing a 1-form on V? A 2-form on V? An n-form on V?

3. What is an *alternating* [multilinear] form on V?
   What happens to the value of an alternating form if we exchange two of its arguments? Why?
   Answer the questions of #2, this time for alternating forms.

4. Suppose that V is a vector space with basis $\mathscr{B} = (\,\vec{b}_1, \vec{b}_2\,)$, and that Q is an alternating 2-form on V.
   Simplify $Q(\alpha_{11}\vec{b}_1 + \alpha_{21}\vec{b}_2,\, \alpha_{12}\vec{b}_1 + \alpha_{22}\vec{b}_2)$. What happens if we switch the order of the subscripts on the α_{ij}'s?

5. Suppose that V is a vector space with basis $\mathscr{B} = (\,\vec{b}_1, \vec{b}_2, \vec{b}_3\,)$, and that T is an alternating 3-form on V.
   Simplify $T(\alpha_{11}\vec{b}_1 + \alpha_{21}\vec{b}_2 + \alpha_{31}\vec{b}_3,\, \alpha_{12}\vec{b}_1 + \alpha_{22}\vec{b}_2 + \alpha_{32}\vec{b}_3,\, \alpha_{13}\vec{b}_1 + \alpha_{23}\vec{b}_2 + \alpha_{33}\vec{b}_3)$.
   What happens if we switch the order of the subscripts on the α_{ij}'s?

. . . and the determinant

Suppose that A and B are $n \times n$ matrices.

6. What three properties characterize the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$?
   Briefly explain why these properties uniquely determine the function.

7. Define the *transpose*, A^{T}, of an $n \times n$ matrix A.
   What is the relationship between $\det A$ and $\det A^{\mathrm{T}}$? Why?

8. Briefly explain the method of "expansion by minors" to compute matrix determinants.
   How is this method justified by the properties of the determinant?

9. What effect does each of the following basic matrix operations have on the determinant of a matrix A? Why?
   (a) Interchanging two columns (or rows) of A.
   (b) Scaling a column (or row) of A by a scalar α.
   (c) Adding a multiple of one column (or row) to another?

10. What shortcuts do we have for determinants of 1×1, 2×2, and 3×3 matrices? Do these work for larger matrices?

*11. How can we use determinants and minors to compute the inverse of an invertible matrix? Why does this work?

12. Justify the following properties of the determinant:
    (a) The determinant of a diagonal matrix is the product of its diagonal entries.
    (*b) $\det(AB) = (\det A)(\det B)$.
    (c) If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.
    (d) $\det(\alpha A) = \alpha^n \det A$.
    (e) If $A \sim B$, then $\det A = \det B$.

The determinant and singularity

1. What does it mean for a square matrix A to be *nonsingular*? *Singular*?

2. In light of our basic row operations and their effect on the determinant, what does the singularity or nonsingularity of A tell us about the number of pivots it gives when reduced?

3. What does the singularity or nonsingularity of an $n \times n$ matrix A allow us to "determine" about:
   (a) The linear transformation A?
   (b) The columns of A?
   (c) The kernel and/or image of A?
   (d) The rank and/or nullity of A?

4. Prove the following statements about singularity of two $n \times n$ matrices A and B:
   (a) If A is nonsingular and B is nonsingular, then AB is nonsingular.
   (b) If AB is singular, then A is singular or B is singular.
   (c) If A is singular or B is singular, then AB is singular.

Eigenvalues and diagonalization II: the characteristic polynomial

Let A be an $n \times n$ matrix.

5. How do we define the *characteristic polynomial* of A?

6. What do the roots of the characteristic polynomial tell us about the matrix A? Why?
   What role does the *field of scalars* play in diagonalizability?

7. By what method can we, in principle, diagonalize any diagonalizable square matrix (or determine that one is not diagonalizable)?
   How can this method be applied to endomorphisms of finite-dimensional vector spaces?

8. For each matrix below, either diagonalize the matrix or determine that it is not diagonalizable:
   (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 13 & 18 \\ -8 & -11 \end{bmatrix}$ (c) $\begin{bmatrix} 32 & -18 \\ 45 & -25 \end{bmatrix}$
   (d) $\begin{bmatrix} 2 & -2 & 0 \\ -24 & 8 & 8 \\ 36 & -16 & -10 \end{bmatrix}$ (*e) $\begin{bmatrix} -4 & -3 & 5 & -2 \\ -11 & -8 & 13 & -6 \\ -7 & -5 & 8 & -4 \\ 8 & 6 & -10 & 4 \end{bmatrix}$

The dot product and norm in $\mathbb{R}^n$

1. Suppose that $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$.
   (a) How do we define the *dot product* of $\vec{x}$ and $\vec{y}$? How is it denoted?
   (b) What is the intuitive interpretation of the value $\vec{x} \cdot \vec{y}$?
   (c) What properties does the dot product have with respect to its parameters?
   (d) For any i, what is $\vec{e}_i \cdot \vec{e}_i$? If $i \neq j$, what is $\vec{e}_i \cdot \vec{e}_j$?

2. Let $\vec{x} \in \mathbb{R}^n$.
   (a) How do we define the *norm* of $\vec{x}$? How is it denoted?
   (b) What is the intuitive interpretation of the value $\|\vec{x}\|$?
   (c) What properties does the norm on $\mathbb{R}^n$ have? Whence do they come?
   (d) For any i, what is $\|\vec{e}_i\|$?

3. Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$, and $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.
   (a) Compute all dot products among pairs of these vectors.
      (i) Which of these pairs of vectors "agrees" most strongly?
      (ii) Which of these pairs of vectors "disagrees" most strongly?
      (iii) Which of these pairs of vectors are geometrically independent of one another?
   (b) Compute the norms of these vectors.
      (i) Which of these vectors is the longest?
      (ii) Which of these vectors is the shortest?

Inner product spaces

4. What is an *inner product* $\langle \cdot, \cdot \rangle$ on a real vector space V? What is an *inner product space*?

5. For any nontrivial closed interval $[a, b] \subset \mathbb{R}$, how can we define an inner product on $\mathrm{C}([a, b])$?

6. How can we define a *norm* on an inner product space? What properties does a norm have?
   How do we interpret the quantity $\|\vec{v} - \vec{w}\|$?

7. In addition to linear combinations, what intrinsically *geometric* notions have meaning in an inner product space? How does this extend our ability to answer linear combination problems?

8. If V and W are inner product spaces, what does it mean for $\phi : V \to W$ to be an *isometry*?

9. Explain the following important analogy:
   set : bijection :: vector space : isomorphism :: inner product space : isometry

The Cauchy-Schwarz inequality and vector projection

Suppose that V is an inner product space and that $\vec{v}, \vec{w} \in V$.

1. Define a function $f(\alpha) \stackrel{\text{def}}{=} \|\vec{v} - \alpha\vec{w}\|^2$, where α is a scalar.
   (a) For each scalar α, what does this function tell us?
   (b) A priori, for any α, what can be said about the value $f(\alpha)$? Why?
   (c) On the other hand, if we expand this quantity in terms of inner products, what do we obtain?
   (d) Combining parts (b) and (c) and using the *discriminant*, what basic property of inner products do we obtain, and what is it called?
   (e) Assume that $\vec{w} \neq \vec{0}$.
   How can we use basic calculus to determine the value of α that minimizes f? What is it?
   (f) What does your answer to part (e) give us? Use it to determine the formula for $\operatorname{proj}_{\vec{w}} \vec{v}$.

2. If $\vec{v}, \vec{w} \neq \vec{0}$, how do we define $\cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$?
   How does the Cauchy-Schwarz inequality help to ensure that such an angle θ exists?

3. Norm inequalities:
   (a) Use the Cauchy-Schwarz Inequality and properties of the inner product to prove the *Triangle Inequality* for inner product spaces: $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$. [Hint: Square both sides first.]
   (b) Use the Triangle inequality to prove that $\|\vec{v} + \vec{w}\| \geq \big|\, \|\vec{v}\| - \|\vec{w}\| \,\big|$.

4. Suppose that f and g are continuous functions on the interval $[0, 1]$. Prove the following:
   (a) $(\int_0^1 f(x)g(x)\,dx)^2 \leq (\int_0^1 f(x)^2\,dx)(\int_0^1 g(x)^2\,dx)$ (b) $\int_0^1 x\,f(x)\,dx \leq \sqrt{\frac{1}{3}\int_0^1 f(x)^2\,dx}$
   (c) $\sqrt{\int_0^1 (f(x) + g(x))^2\,dx} \leq \sqrt{\int_0^1 f(x)^2\,dx} + \sqrt{\int_0^1 g(x)^2\,dx}$.

5. Let $\vec{a} \in V$ with $\vec{a} \neq \vec{0}$, and define $L : V \to V$ by $\vec{v} \mapsto \operatorname{proj}_{\vec{a}} \vec{v}$. Show that:
   (a) L is a linear transformation. (b) $L \circ L = L$.

6. Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. Compute the following:
   (a) $\operatorname{proj}_{\vec{a}} \vec{b}$ (b) $\operatorname{proj}_{\vec{b}} \vec{a}$ (c) $\operatorname{proj}_{\vec{b}} \vec{c}$ (d) $\operatorname{proj}_{\vec{c}} \vec{b}$
   (e) $\operatorname{proj}_{\vec{a}} (\operatorname{proj}_{\vec{b}} \vec{c})$ (f) $\operatorname{proj}_{\vec{b}} (\operatorname{proj}_{\vec{a}} \vec{c})$

7. Consider the functions $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$ in $\mathrm{C}([0, 1])$ (with the usual inner product).
   Compute the following:
   (a) $\|f\|$ (b) $\|g\|$ (c) $\|h\|$
   (d) $\operatorname{proj}_f g$ (e) $\operatorname{proj}_g f$ (f) $\operatorname{proj}_g h$ (g) $\operatorname{proj}_h g$

8. Consider the inner product space $\mathrm{C}([-\pi, \pi])$ with $\langle f, g\rangle \stackrel{\text{def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} fg$, and let $s_k(x) = \sin kx$ and $f(x) = x$.
   Compute $\operatorname{proj}_{s_1} f$, $\operatorname{proj}_{s_2} f$, and $\operatorname{proj}_{s_3} f$; in general, what is $\operatorname{proj}_{s_k} f$ for other whole numbers k?

Orthogonality, orthogonal projection, and the Gram-Schmidt process: in principle

Let V be an inner product space.

1. What does it mean for two vectors $\vec{v}, \vec{w} \in V$ to be *orthogonal*? How do we denote this?

2. Unit vectors:
   (a) What does it mean for $\vec{w} \in V$ to be a *unit vector*?
   (b) How can we turn any nonzero vector $\vec{v}$ into a unit vector in the same direction? Verify directly.
   (c) How does the formula for $\operatorname{proj}_{\vec{w}} \vec{v}$ simplify if $\vec{w}$ is a unit vector?

3. What is an *orthonormal* collection of vectors?

4. Briefly explain the Gram-Schmidt orthogonalization process; include what it takes as input, what it produces as output, and the steps in the process.

5. Suppose that $\mathscr{B} = (\, \vec{v}_1, \, \vec{v}_2, \, \ldots, \, \vec{v}_n \,)$ is an *orthonormal basis* for V.
   (a) Conceptually, why are orthonormal bases the only "good" bases to use for an inner product space? Explain in terms of the isomorphisms $[\mathscr{B}]$ and $[\mathscr{B}]^{-1}$.
   (b) How can we easily compute $[\mathscr{B}]^{-1}\vec{v}$ for a vector $\vec{v} \in V$?
   (c) How can we find an orthonormal basis for any finite-dimensional inner product space?

. . . and in practice

6. Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \not\perp \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$,
   and change each of these vectors into a unit vector in the same direction.

7. Prove that $\vec{v} \perp \vec{w} \;\Leftrightarrow\; \|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$.

8. Let $\vec{v}, \vec{w} \in V$ with $\vec{w} \neq \vec{0}$, and let $\operatorname{orth}_{\vec{w}} \vec{v} = \vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}$. Prove that $\operatorname{orth}_{\vec{w}} \vec{v} \perp \vec{w}$.

9. Suppose that $\vec{v} \in V$ and that $\mathscr{C} = \{\, \vec{v}_1, \, \vec{v}_2, \, \ldots, \, \vec{v}_k \,\}$ is an orthonormal collection in V.
   Let $\vec{v}\,' = \vec{v} - \langle \vec{v}, \vec{v}_1 \rangle \, \vec{v}_1 - \langle \vec{v}, \vec{v}_2 \rangle \, \vec{v}_2 - \cdots - \langle \vec{v}, \vec{v}_k \rangle \, \vec{v}_k$.
   (a) Prove that $\forall\, i, \; \vec{v}\,' \perp \vec{v}_i$.
   (b) Prove that $\forall\, \vec{w} \in \operatorname{span} \mathscr{C}, \; \vec{v}\,' \perp \vec{w}$.

10. Prove that if $\mathscr{C} = \{\, \vec{v}_1, \, \ldots, \, \vec{v}_n \,\}$ is an orthonormal collection, then $\mathscr{C}$ is linearly independent.

11. Let $\mathscr{B} = \{\, \frac{1}{\sqrt{2}}, \; \cos x, \; \sin x, \; \cos 2x, \; \sin 2x \,\} \subset \mathrm{C}([-\pi, \pi])$, with inner product $\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} fg$.
    (a) Show that $\mathscr{B}$ is an orthonormal collection.
    (b) Suppose that $f(x) \in \operatorname{span} \mathscr{B}$ and that $f(x) = \frac{\alpha_0}{\sqrt{2}} + \alpha_1 \cos x + \alpha_2 \sin x + \alpha_3 \sin 2x + \alpha_4 \sin 2x$.
    Find integral formulæ for each of the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_4$.

12. Find an orthonormal basis for $\operatorname{span}\{\, 1, \; x, \; x^2, \; x^3 \,\} \subset \mathrm{C}([0,1])$ (with respect to the usual inner product).

13. Find an orthonormal basis for $\operatorname{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \subset \mathbb{R}^3$ (with respect to the usual inner product).

14. Find an orthonormal basis for $\operatorname{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^3$ (with respect to the usual inner product).

Orthonormal bases, projection, and approximation

Suppose that V is an inner product space and that $W \subset V$ is a subspace with orthonormal basis $\mathscr{B} = (\,\vec{w}_1,\ \vec{w}_2,\ \ldots,\ \vec{w}_n\,)$.

1. We say that a vector $\vec{n} \in V$ is orthogonal to the *subspace* W (denoted $\vec{n} \perp W$) when $\forall\, \vec{w} \in W,\ \vec{n} \perp \vec{w}$.
   Show that if $\vec{n} \perp W$, then $\forall\, \vec{w} \in W,\ \|\vec{n} + \vec{w}\| \geq \|\vec{n}\|$.

2. Suppose that $\vec{v} \in V$.
   (a) If $\vec{v} \in V$, by what formula (involving $\mathscr{B}$) do we define $\operatorname{proj}_W \vec{v}$?
   (b) Define $\operatorname{orth}_W \vec{v} = \vec{v} - \operatorname{proj}_W \vec{v}$ (note that $\vec{v} = \operatorname{proj}_W \vec{v} + \operatorname{orth}_W \vec{v}$).
   What does Problem Set 23, #9 tell us about the relationship between $\operatorname{orth}_W \vec{v}$ and W?
   (c) Combine part (b) and the previous problem to show that $\forall\, \vec{w} \in W,\ \|\vec{v} - \vec{w}\| \geq \|\vec{v} - \operatorname{proj}_W \vec{v}\|$.
   What does this tell us about the vector $\operatorname{proj}_W \vec{v} \in W$?

3. Let $\mathscr{F}_n = \{\, \frac{1}{\sqrt{2}},\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos nx,\ \sin nx \,\} \subset \mathrm{C}([-\pi, \pi])$,
   with inner product defined by $\langle f, g\rangle \stackrel{\text{def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} fg$ (recall that $\mathscr{F}_n$ is then an orthonormal collection).
   Given $f(x) \in \mathrm{C}([-\pi, \pi])$, consider the n^{th}-order *Fourier polynomial*, $F_n(x) \stackrel{\text{def}}{=} \operatorname{proj}_{\operatorname{span} \mathscr{F}_n} f(x)$:
   (a) By what formula can we compute $F_n(x)$?
   (b) What sort of function is $F_n(x)$? How does it relate to $\mathscr{F}_n$ and the function $f(x)$?
   (c) What do we expect to happen with $F_n(x)$ as n increases? Why?
   Briefly explain how this is all simply an application of the general theory of orthogonal projection onto the span of an orthonormal collection in an inner product space.

...and the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Consider the function $f(x) = x \in \mathrm{C}([-\pi, \pi])$, with the usual inner product and orthonormal Fourier basis.
Answer the following questions regarding the expression of $f(x)$ as a Fourier polynomial:

4. Find $\left\langle x, \frac{1}{\sqrt{2}} \right\rangle$.

5. For $k > 0$, find $\langle x, \cos kx\rangle$. [Hint: You're integrating an odd function over $[-\pi, \pi]$.]

6. For $k > 0$, find $\langle x, \sin kx\rangle$. [Hint: $\int x \sin kx\, dx = \frac{1}{k^2}\sin kx - \frac{1}{k}x\cos kx + C$.]

7. Find the fourth-order Fourier polynomial $F_4(x)$ for $f(x)$.

8. Consider the n^{th}-order Fourier polynomial $F_n(x)$ for $f(x)$.
   (a) Find a formula for $F_n(x)$.
   (b) Compute $\|F_n(x)\|^2$ [Hint: Remember that $F_n(x)$ is expressed via an orthonormal basis.]
   (c) If we take $\lim\limits_{n\to\infty} \|F_n(x)\|^2$, what series results?
   (d) On the other hand, what results if we compute $\|f(x)\|^2$ directly?
   (e) Assuming that your answers in parts (c) and (d) match, find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Dot products and the transpose in $\mathbb{R}^m$

1. How do we transpose a column vector $\vec{x} \in \mathbb{R}^m$?
   If $\vec{x}, \vec{y} \in \mathbb{R}^m$, how can we use transposes to compute the dot product $\vec{x} \cdot \vec{y}$?

2. Suppose that A is an $m \times n$ matrix.
   (a) How do we define the *transpose* of the matrix A? How is it denoted?
   What should we view a transposed matrix to be split into?
   (b) How does the transpose behave with respect to addition and scaling of matrices?
   (c) How does the transpose relate to matrix composition?

3. If A is an $m \times n$ matrix and $\vec{y} \in \mathbb{R}^m$, what do the entries of $A^{\mathrm{T}}\vec{y}$ represent? Why?
   Use this to show that $\vec{y} \perp C(A) \;\Leftrightarrow\; A^{\mathrm{T}}\vec{y} = \vec{0}$.

4. If A and B are $m \times n$ matrices, what do the entries of $A^T B$ represent? Why?
   If the columns of an $m \times n$ matrix A form an orthonormal collection, what does $A^{\mathrm{T}}A$ equal?

Projection matrices and approximation for linear systems: in principle

Suppose that a $m \times n$ matrix A has linearly independent columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$.

5. What is the size of the matrix $A^{\mathrm{T}}A$? Show that if $\vec{x} \in \mathbb{R}^n$ and $A^{\mathrm{T}}A\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$.
   What does this allow us to conclude about the linear transformation $A^{\mathrm{T}}A$?

6. Consider the linear transformation given by the projection matrix $P = A\,(A^T A)^{-1} A^T$.
   (a) What are the domain and codomain of the linear transformation given by P?
   (b) Show that if $\vec{y} \in \mathbb{R}^m$ and $\vec{y} \perp C(A)$, then $P\vec{y} = \vec{0}$.
   (c) Show that $\forall\, \vec{y} \in C(A),\; P\vec{y} = y$.
   (d) Show that if $\vec{p} \in C(A)$ and $\vec{n} \perp C(A)$, then $P(\vec{p} + \vec{n}) = \vec{p}$.
   Let $\vec{v} \in \mathbb{R}^m$, $\vec{p} = \mathrm{proj}_{C(A)}\,\vec{v}$, and $\vec{n} = \mathrm{orth}_{C(A)}\,\vec{v}$ in the above to show that $P\vec{v} = \mathrm{proj}_{C(A)}\,\vec{v}$.

7. In the case that the columns of A are *orthonormal*, how does the above formula for P simplify? Why?
   What if the columns of A form a basis for $\mathbb{R}^m$?

8. Given the [possibly inconsistent] linear system $[\,A \mid \vec{b}\,]$, how can we use orthogonal projection to find the best [possibly approximate] solution $\vec{x}$ to this system?
   How does this allow us to find lines or curves that “best fit” some set of data points?

. . . and in practice

9. Find the line $y = ax + b$ that best fits the points $\{\,(1,1),\ (5,2),\ (7,3),\ (8,4),\ (15,10)\,\}$.

10. Find the parabola $y = ax^2 + bx + c$ that best fits the points $\{\,(0,0),\ (0,1),\ (3,3),\ (5,3),\ (6,-1)\,\}$.

11. Find the plane $z = ax + by + c$ that best fits the points $\{\,(0,0,0),\ (1,2,2),\ (3,2,1),\ (4,2,1)\,\}$.

12. Find the exponential curve $y = Ae^{kx}$ that best fits the points $\{\,(0,1),\ (2,e),\ (4,e^{3/2}),\ (5,e^2)\,\}$.
    [Hint: Take the natural log of both sides of the equation, then set up a linear system to solve for $\underline{\ln A}$ and k.]