Linear combinations of matrices

- Let A and B be $m \times n$ matrices.
  - Scalar multiplication αA defines an $m \times n$ matrix by scaling each entry of A by α.
  - Matrix addition $A + B$ defines an $m \times n$ matrix by adding corresponding entries of the matrices A and B.
    - The *zero matrix*, 0, is the matrix whose entries are all zero.
    - The *negative*, $-A$, of a matrix A is the matrix whose entries are the negatives of those in A.
- These operations satisfy all of the usual rules:

$$\begin{aligned} A + B &= B + A & 1A &= A \\ (A+B)+C &= A+(B+C) & \alpha(A+B) &= \alpha A + \alpha B \\ 0 + A = A &= A + 0 & (\alpha+\beta)A &= \alpha A + \beta A \\ A + (-A) = 0 &= (-A) + A & (\alpha\beta)A &= \alpha(\beta A) \end{aligned}$$

- The set $M_{m\times n}$ of $m \times n$ matrices thus forms a *vector space* (of dimension mn).

Matrix composition

Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix.

- AB is an $m \times p$ matrix that gives the composition of the linear transformations $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^p \to \mathbb{R}^n$.
- We compute compositions one column at a time: $\boxed{\text{the } j^{\text{th}} \text{ column of } AB = A(\text{the } j^{\text{th}} \text{ column of } B)}$.
- The *identity matrix* of size n is the $n \times n$ matrix $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$, which represents the identity l.t. from $\mathbb{R}^n \to \mathbb{R}^n$.
- Matrix composition satisfies the following rules (for matrices of the appropriate sizes):

$$\begin{aligned} (AB)C &= A(BC) \\ (\alpha A)B = \alpha(AB) &= A(\alpha B) \\ A(B_1 + B_2) &= AB_1 + AB_2 \\ (A_1 + A_2)B &= A_1 B + A_2 B \\ IA = A &= AI \end{aligned}$$

- Matrix composition is *not commutative*: in general, $AB \neq BA$.

The matrix inverse

- A matrix A is *invertible* if there is some matrix, denoted A^{-1}, for which $AA^{-1} = I = A^{-1}A$.
  - In order for A to be invertible, it must give an invertible linear transformation, which must be bijective; thus, A must be square, with zero nullity and full rank.
  - Composing with an invertible matrix's inverse can be used to cancel that matrix from an equation; for example, if A is an invertible matrix, then we can solve $A\vec{x} = \vec{b}$ as $\vec{x} = A^{-1}\vec{b}$.
- For an invertible matrix A, we can compute A^{-1} by row-reducing $[\,A \mid I\,] \rightsquigarrow [\,I \mid A^{-1}\,]$.
  - Why does this work? $AA^{-1} = I \Rightarrow A(A^{-1}\vec{e}_j) = \vec{e}_j$, so we find the j^{th} column of A^{-1} by solving $[\,A \mid \vec{e}_j\,]$.

Matrix minors

- If A is an $n \times n$ matrix, its $(i,j)^{\text{th}}$ *minor*, denoted A_{ij}, is the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of A; this concept is useful for certain determinant-related computations.

The matrix determinant

For each n, there is a unique multilinear, alternating n-form on $\mathbb{R}^n$ for which $I_n \mapsto 1$; it is called the *determinant*.

- This can be considered as a function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ mapping each $n \times n$ matrix to a scalar value called its determinant. $\det$ thus has the following basic properties:
  - $\det$ is multilinear on columns (i.e., with respect to each column separately, $\det$ has the properties of a l.t.).
  - $\det A$ is zero if any two columns of A are equal, so switching two columns negates the value of $\det$.
  - $\det I_n = 1$.
- It happens that $\det A^{\mathrm{T}} = \det A$; thus, each column-property of $\det$ listed above works for *rows*, as well.
- The determinant behaves well with respect to basic row and column operations on a matrix A
  - Exchanging two rows or columns of A negates $\det A$.
  - Scaling a row or column of A by α scales $\det A$ by α.
  - Adding a multiple of one row or column of A to another does not change $\det A$.
- $\det(AB) = (\det A)(\det B)$, which implies:
  - If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$.
  - If $A \sim B$, then $\det A = \det B$.
- The determinant of a matrix A can be computed by the method of *expansion by minors*:
  - Choose a row or column of A to expand upon.
  - Nominally assign each entry in that row or column either a $+$ or a $-$, starting with a $+$ in the top-left entry of the matrix and alternating $+ - + - \cdots$ from it.
  - For each entry in the chosen row or column, multiply:

    the nominal sign of the entry $\times$ the value of the entry $\times$ the determinant of the entry's minor in A;

    their sum gives $\det A$.
- The determinants of small matrices can be computed directly via shortcuts (which do *not* work for larger matrices):
  - $\det[\,a\,] = a$
  - $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
  - $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3$
- We can form the inverse of an $n \times n$ matrix A via determinants (inefficiently), as follows:
  - Form a new $n \times n$ matrix containing the determinants of the corresponding minors of A.
  - Change the signs of all its entries according to the rule for expansion by minors.
  - Transpose.
  - Scale the resulting matrix by $\frac{1}{\det A}$.

  The result is A^{-1}; this can be checked by verifying that the entries of AA^{-1} and $A^{-1}A$ give certain minor expansions evaluating to 1 along the diagonal and 0 elsewhere.

. . . and singularity

An $n \times n$ matrix A is called *singular* if $\det A = 0$; it is called *nonsingular* if $\det A \neq 0$.

- When A is nonsingular: A is bijective and thus invertible; the columns of A form a basis for $\mathbb{R}^n$; $\ker A = \{\vec{0}\}$ and $\operatorname{im} A = \mathbb{R}^n$; nullity $A = 0$ and rank $A = n$.
- When A is singular: A is neither injective nor surjective, and non-invertible; the columns of A are not linearly independent and do not span $\mathbb{R}^n$; $\ker A \neq \{\vec{0}\}$ and $\operatorname{im} A \neq \mathbb{R}^n$; nullity $A > 0$ and rank $A < n$.

. . . and eigenvalues

If A is a square matrix, the *characteristic polynomial* of A is the polynomial in λ defined by $\det(A - \lambda I)$. Key property:

$$\boxed{\lambda \text{ is an eigenvalue of } A \iff \det(A - \lambda I) = 0}.$$

This allows us determine the eigenvalues of *any* square matrix, by finding the roots of its characteristic polynomial.