Linear transformations, kernel, and image — Math 200

Swenton

Linear transformations

Suppose that V and W are vector spaces.

- $L : V \to W$ is a *linear transformation* means:
  $$\boxed{\forall\, \vec{v}_1\ \dots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \dots, \alpha_n,\ L(\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n) = \alpha_1 L\vec{v}_1 + \cdots + \alpha_n L\vec{v}_n}.$$
  - This means that the function L and the operation of forming linear combinations commute (this can be illustrated by a *commutative diagram*).
  - Three-point checklist for L to be a linear transformation:
    - $L(\vec{0}_V) = \vec{0}_W$.
    - $\forall\, \vec{v} \in V$ and scalar α, $L(\alpha\vec{v}) = \alpha L\vec{v}$.
    - $\forall\, \vec{v}_1, \vec{v}_2 \in V$, $L(\vec{v}_1 + \vec{v}_2) = L\vec{v}_1 + L\vec{v}_2$.

Kernel and image

Suppose that $L : V \to W$ is a linear transformation.

- The *kernel* of L is $\ker L = \{\, \vec{v} \in V \ :\ L\vec{v} = \vec{0}_W \,\}$, i.e., $\boxed{\vec{v} \in \ker L \ \Leftrightarrow\ L\vec{v} = \vec{0}}$.
  - $\ker L$ is a *subspace* of the *domain* of L, relating to the *injectivity* of the function L:
    - $L\vec{v}_1 = L\vec{v}_2 \ \Leftrightarrow\ \vec{v}_1 - \vec{v}_2 \in \ker L$.
    - L is injective $\Leftrightarrow$ $\ker L = \{\, \vec{0} \,\}$.
- The *image* of L is $\operatorname{im} L = \{\, L\vec{v} \ :\ \vec{v} \in V \,\}$, i.e., $\boxed{\vec{w} \in \operatorname{im} L \ \Leftrightarrow\ \exists\, v \in V \text{ with } \vec{w} = L\vec{v}}$.
  - $\operatorname{im} L$ is a *subspace* of the *codomain* of L, relating to the *surjectivity* of the function L:
    - L is surjective $\Leftrightarrow$ $\operatorname{im} L = W$.

Rank and nullity

Suppose that $L : V \to W$ is a linear transformation, where V and W are finite-dimensional vector spaces.

- *Nullity*: $\operatorname{nullity} L \overset{\text{def}}{=} \dim(\ker L)$
  - $0 \le \operatorname{nullity} L \le \dim V$, because $\ker L$ is a subspace of V.
  - L is injective $\Leftrightarrow$ $\operatorname{nullity} L = 0$.
- *Rank*: $\operatorname{rank} L \overset{\text{def}}{=} \dim(\operatorname{im} L)$
  - $0 \le \operatorname{rank} L \le \dim W$, because $\operatorname{im} L$ is a subspace of W.
  - L is surjective $\Leftrightarrow$ $\operatorname{rank} L = \dim W$.
- The Rank+Nullity Theorem: $\boxed{\operatorname{rank} L + \operatorname{nullity} L = \dim V}$.
  - Outline of proof: take a basis for $\ker L$, and extend it to a basis for V; apply L to the non-kernel basis vectors above and show that these form a basis for $\operatorname{im} L$, then count the vectors in the bases to determine dimensions.

Linear transformations from collections and from matrices

- A collection $\mathscr{C} = (\, \vec{v}_1,\, \vec{v}_2,\, \dots,\, \vec{v}_n \,)$ of vectors in V defines a linear transformation $[\mathscr{C}] : \mathbb{R}^n \to V$,

  defined by forming linear combinations: $$\boxed{[\mathscr{C}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{def}}{=} x_1\vec{v}_1 + \cdots + x_n\vec{v}_n}.$$
  - $\ker[\mathscr{C}] \leftrightarrow$ (coefficients for) linear relations on $\mathscr{C}$ (so $[\mathscr{C}]$ is injective $\Leftrightarrow \mathscr{C}$ is linearly independent in V).
  - $\operatorname{im}[\mathscr{C}] = \operatorname{span}\mathscr{C}$ (so $[\mathscr{C}]$ is surjective $\Leftrightarrow \mathscr{C}$ spans V). (Thus, $[\mathscr{C}]$ is bijective $\Leftrightarrow \mathscr{C}$ is a basis for V.)
- An $m \times n$ matrix A gives a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ defined by forming linear combinations of its *columns*:
  $$\boxed{A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \overset{\text{def}}{=} x_1[1^{\text{st}} \text{ col. of } A] + \cdots + x_n[n^{\text{th}} \text{ col. of } A]}.$$
  - $\ker A$ is the set of solutions to the homogeneous system $[\, A \mid \vec{0}\,]$ (also known as $\mathrm{N}(A)$, the *nullspace* of A).
    - We can find a *basis* for $N(A)$ simply by finding the free variables' contributions to the solution of the homogeneous system $[\, A \mid \vec{0}\,]$.
  - $\operatorname{im} A$ is the span of the columns of A (also known as $\mathrm{C}(A)$, the *column space* of A).
    - We can find a *basis* for $C(A)$ simply by reducing A and taking the columns of A that gave pivots.
  - Matrices and the standard basis: $\boxed{A\vec{e}_j = j^{\text{th}} \text{ column of } A}$.