The FTLT and change of basis

Matrix representation of linear transformations

- Theorem: Every linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is given by a matrix.
  - Proof: Let A be the $m \times n$ matrix whose j^{th} column is $L\vec{e}_j$, and show that $\forall\, \vec{x} \in \mathbb{R}^n$, $A\vec{x} = L\vec{x}$.
- **The Fundamental Theorem of Linear Transformations:** Every linear transformation $L : V \to W$, where V and W are finite-dimensional vector spaces, can be represented by a matrix.
  - Proof: · Let $\mathscr{V} = (\, \vec{v}_1, \ldots, \vec{v}_n \,)$ be a basis for V and $\mathscr{W} = (\, \vec{w}_1, \ldots, \vec{w}_m \,)$ be a basis for W.
    · Use the isomorphisms provided by the bases $\mathscr{V}$ and $\mathscr{W}$ to form ${}_{\mathscr{W}}[L]_{\mathscr{V}} = [\mathscr{W}]^{-1} L\, [\mathscr{V}]$; since ${}_{\mathscr{W}}[L]_{\mathscr{V}} : \mathbb{R}^n \to \mathbb{R}^m$, it is given by a matrix.
    · Solving, $L = [\mathscr{W}]_{\mathscr{W}}[L]_{\mathscr{V}}[\mathscr{V}]^{-1}$.
  - The workings of the FTLT are illustrated by a *commutative diagram*.
  - With L, $\mathscr{V}$, and $\mathscr{W}$ as above, ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ can thus be computed as follows:

    The j^{th} column of ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ is the $\mathscr{W}$-coordinate vector of $L\vec{v}_j$.
  - A matrix representation of a linear transformation L is *not intrinsic* to L—the matrix depends very much upon the choice of bases for the domain and codomain.
- Matrix representations allow us, after choosing bases for the domain and codomain of a linear transformation L, to translate any computation involving L into a question about matrices and column vectors; once these simpler questions are answered, we can use our bases to translate our answers back to their original context.

Change of basis

- Given $L : V \to W$, many different matrices can be used to represent L, depending on our choice of bases for V and W. If we have bases $\mathscr{V}$ and $\mathscr{V}'$ for V and bases $\mathscr{W}$ and $\mathscr{W}'$ for W, the matrices ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ and ${}_{\mathscr{W}'}[L]_{\mathscr{V}'}$ are related by:
$$ {}_{\mathscr{W}'}[L]_{\mathscr{V}'} = (\underline{[\mathscr{W}']^{-1}[\mathscr{W}]})_{\mathscr{W}}[L]_{\mathscr{V}}(\underline{[\mathscr{V}]^{-1}[\mathscr{V}']}); $$
  The underlined matrices above are called change-of-basis matrices.
  - Attaching the diagrams for ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ and ${}_{\mathscr{W}'}[L]_{\mathscr{V}'}$ gives us a *commutative diagram* illustrating change of basis.
- In general, given bases $\mathscr{A}$ and $\mathscr{B}$ for a k-dimensional vector space, the isomorphism $[\mathscr{B}]^{-1}[\mathscr{A}] : \mathbb{R}^k \to \mathbb{R}^k$ is given by an invertible $k \times k$ matrix, called a *change-of-basis matrix*.
  - $[\mathscr{B}]^{-1}[\mathscr{A}]$ sends each $\mathscr{A}$-coordinate vector to the $\mathscr{B}$-coordinates for the same vector, so:

    The j^{th} column of $[\mathscr{B}]^{-1}[\mathscr{A}]$ is the $\mathscr{B}$-coordinate vector of the j^{th} vector of $\mathscr{A}$.
  - In the case of column vectors, we can use matrix reduction to compute change-of-basis matrices:
$$ [\,\mathscr{B} \mid \mathscr{A}\,] \rightsquigarrow \left[\, I \mid \underline{[\mathscr{B}]^{-1}[\mathscr{A}]} \,\right]. $$
    We can also use this method to compute change-of-basis matrices for other types of vectors, by using a basis to translate the given bases $\mathscr{A}$ and $\mathscr{B}$ into column vectors.
- Via change of basis, the question of whether two matrices A and B are *equivalent* (i.e., represent the same linear transformation, possibly with respect to different bases) amounts to whether $\exists$ invertible matrices X and Y such that $B = XAY^{-1}$.
  - Taking bases $\mathscr{V}$ and $\mathscr{W}$ for the domain and codomain of $L : V \to W$, respectively, along the lines of those in the proof of the Rank+Nullity Theorem, we can force ${}_{\mathscr{W}}[L]_{\mathscr{V}}$ into the form
$$ \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ & 0 & & 0 \end{bmatrix}, \text{ where the matrix is of size } \dim W \times \dim V \text{ and has } (\operatorname{rank} L) \text{ ones.} $$
    Any matrix A is equivalent to one, and only one, such matrix, giving a simple resolution of the equivalence problem: A and B are represent equivalent linear transformations if, and only if, they have the same *size* and *rank*.
- Each of our basic *row operations* for matrix reduction (exchange, scaling, and adding multiples) is the effect of a particular *change of basis*.
  - This provides a conceptual explanation of our method of matrix reduction for solving linear systems: we are changing the basis in which we represent our vectors, until the linear combination problem becomes simple enough to see the answer.