Name: ______________________________

Exam 1: Math 200A, Fall 2021

Swenton

This exam consists of six problems, each on its own page. Show all work neatly and circle your answers; answers given without the work justifying them will not receive full credit. Be sure to answer each question as asked; answers need only be simplified when the problem states so.
NO CALCULATORS are allowed on this examination.
Please work carefully and keep your work neat and organized. Though this is set as a 90-minute exam, you will be allowed extra time if necessary.
— May you be at your best. Don't panic! And good luck.

> All problems on this exam refer to *real* vector spaces, i.e., vector spaces over the field $\mathbb{R}$.

After you finish, you must *write* and *sign* the honor pledge:

I have neither given nor received unauthorized aid on this examination.

1.	/20	4.	/20
2.	/20	5.	/20
3.	/20	6.	/20

Total: / 120

1. (20 points) Consider the following linear system:

$$\begin{aligned} a + b + c + 14e &= 2d + 7 \\ b + 13e &= 2c + 2d + 5 \\ 2a + 2b + 2c + 24e &= 3d + 12 \end{aligned}$$

(a) Write this linear system in augmented-matrix form (arranging the variables in alphabetical order):

(b) The matrix in part (a) row-reduces to presented to the right. Circle the *pivots*, and mark the columns for *free variables* as **FREE**.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & -4 & -2 \end{array}\right]$$

(c) Form a *solution vector* for this system, and split off the free variables' contributions; circle the solution to the corresponding *homogeneous* system.

Solution vector: $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} =$

2. (20 points) In each part below, consider the given collection of vectors, which is given as having the underlined property.

For each modification given, determine what can be determined about the new collection:
YES if it must have that property; **NO** if it can't; or **MAYBE** if it might or might not.

Briefly explain each answer (just a phrase or sentence suffices).

(a) Suppose the collection $\mathscr{D} = \{ \vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4 \}$ spans the vector space V.

(i) $\{ \vec{w}_1, \vec{w}_2, \vec{w}_4 \}$

(ii) $\{ \vec{w}_4, \vec{w}_2, \vec{w}_3, \vec{w}_1 \}$

(iii) $\{ \vec{w}_1, \vec{w}_2, \vec{w}_3 - 5\vec{w}_1, \vec{w}_4 \}$

(iv) $\{ \vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{0} \}$

(b) Suppose the collection $\mathscr{C} = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is linearly independent in the vector space V.

(i) $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_1 + \vec{v}_2 \}$

(ii) $\{ \vec{v}_3, \vec{v}_1, \vec{v}_2 \}$

(iii) $\{ \vec{v}_1, \vec{v}_3 \}$

(iv) $\{ \vec{v}_1, \vec{v}_1 + 2\vec{v}_2 - \vec{v}_3, \vec{v}_3 \}$

3. (20 points) Consider the vector space $\mathbb{R}[x]$ of all real polynomials in the variable x.

One of the two subsets below is a subspace of $\mathbb{R}[x]$, and the other is not; prove that the one that is a subspace is, indeed, a subspace, and show explicitly that the other is not:

$$U = \{\, A + Bx + Cx^2 \ : \ A, B, C \leq 1 \,\}$$

$$V = \{\, A + Bx + Cx^2 \ : \ 3B + 8C = A \,\}$$

4. (20 points) Prove the following assertion *directly*, i.e., using only definitions and basic logic. Be sure that your steps are justified and properly address the definitions used.

Claim: If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans V,
then $\{\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3, \vec{v}_2, \vec{v}_3\}$ spans V.

Proof:

5. (20 points) Prove the following assertion *directly*, i.e., using only definitions and basic logic. Be sure that your steps are justified and properly address the definitions used.

Claim: If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent,
then $\{\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Proof:

6. (20 points) Consider the problem of building the vector $\vec{w} \in \mathbb{R}^4$ as a linear combination of the collection $\mathscr{C} = \{\, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \,\}$. The augmented matrix below is the result of row-reducing $[\,\mathscr{C} \,|\, \vec{w}\,]$; use it to answer the questions below (note that your answers will be in terms of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, and $\vec{w}$, as you don't have enough information to know what, exactly, they equal).

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & 0 & -2 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Write $\vec{w}$ as as a linear combination of $\mathscr{C}$:

$\vec{w} =$

(b) Does the collection $\mathscr{C}$ span $\mathbb{R}^4$? Why or why not?

(c) Find a nontrivial linear relation on the collection $\mathscr{C}$.
What does this tell us about the collection?

(d) We wish to remove *one vector* from $\mathscr{C}$ in order to make it linearly independent.
In light of your answer above and/or the row-reduced matrix, explain which vector(s) would achieve this and which vector(s) would not.