Endomorphisms, similarity, and diagonalization **Math 200**

Swenton

Let $L : V \to V$ be an *endomorphism*, i.e., a linear transformation whose domain and codomain are the same vector space.

- We can take *powers* of an endomorphism by composing it with itself: $L^k \stackrel{\text{def}}{=} \underbrace{L \circ L \circ \cdots \circ L}_{k \text{ times}}$.
  - By convention, L^0 is the *identity* endomorphism $\vec{v} \mapsto \vec{v}$.
  - In the case of an *automorphism* (i.e., an invertible endomorphism), we can also define negative powers, via $L^{-k} \stackrel{\text{def}}{=} \underbrace{L^{-1} \circ L^{-1} \circ \cdots \circ L^{-1}}_{k \text{ times}}$.
- Two endomorphisms $A, B : V \to V$ *commute* if $AB = BA$.
  Endomorphisms *do not*, in general, commute (nor do other linear transformations and matrices)! Thus, order of composition matters, and when composing with a linear transformation or matrix, we must specify on which side we're composing.

The FTLT and change of basis *for endomorphisms*

- In the FTLT, we use the *same* basis $\mathscr{V}$ for the domain and the codomain, and we shorten ${}_{\mathscr{V}}[L]_{\mathscr{V}}$ to $[L]_{\mathscr{V}}$.
  - Because of this, matrix representations respect powers:
    - $[L^k]_{\mathscr{V}} = ([L]_{\mathscr{V}})^k$.
    - If L is invertible, then $[L^{-1}]_{\mathscr{V}} = ([L]_{\mathscr{V}})^{-1}$.
- Change of basis, then, amounts to the choice of a second basis $\mathscr{V}'$ and $[L]_{\mathscr{V}'} = (\underline{[\mathscr{V}']^{-1}[\mathscr{V}]})[L]_{\mathscr{V}}(\underline{[\mathscr{V}]^{-1}[\mathscr{V}']})$.
  - The change-of-basis matrices for the domain and codomain are *inverses* of one another; thus, the equivalence problem becomes whether $B = XAX^{-1}$ for some invertible matrix X.
- At the level of matrices, the equivalence problem for endomorphisms is that of *similarity*:
  - Two square matrices A and B are *similar*, written $A \sim B$, if $\exists$ an invertible matrix X such that $B = XAX^{-1}$.
  - Two matrices are similar just when they are both matrices for the same endomorphism (possibly with respect to different bases).
- An endomorphism $L : V \to V$ is *diagonalizable* if $\exists$ some basis $\mathscr{V} = (\,\vec{v}_1, \ldots, \vec{v}_n\,)$ for V such that $[L]_{\mathscr{V}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.
  Such a *diagonalization* of L allows us to very simply describe the action of the endomorphism L: $L\vec{v}_j = \lambda_j \vec{v}_j$.
  - In the case of a matrix A, this amounts to A being *similar to a diagonal matrix* Λ, i.e., $A = X\Lambda X^{-1}$.

Eigenvalues and eigenvectors

Suppose that $L : V \to V$ is an endomorphism, where V is n-dimensional.

- A nonzero vector $\vec{v} \in V$ *is an eigenvector of L with eigenvalue* λ means: $\boxed{L\vec{v} = \lambda\vec{v}}$.
  - A nonzero vector $\vec{v} \in V$ *is an eigenvector of* L means: $\exists$ some scalar λ with $L\vec{v} = \lambda\vec{v}$.
  - A scalar λ *is an eigenvalue of* L means: $\exists$ some nonzero vector $\vec{v} \in V$ with $L\vec{v} = \lambda\vec{v}$.
- For a given scalar λ, the *eigenspace* of λ is the subspace $E_\lambda \stackrel{\text{def}}{=} \{\, \vec{v} \in V : L\vec{v} = \lambda\vec{v} \,\}$ containing $\vec{0}$ and all eigenvectors for λ.
  - A nonzero vector $\vec{v} \in V$ is an eigenvector of L with eigenvalue $\lambda \iff \vec{v} \in E_\lambda$.
  - λ is an eigenvalue of $L \iff E_\lambda \neq \{\,\vec{0}\,\} \iff \dim E_\lambda > 0$.
- To determine a maximal collection of linearly independent eigenvectors for a given eigenvalue λ of a *matrix* A, find a basis for $N(A - \lambda I)$, i.e., take the free variables' contributions to the solution of the homogeneous system $[\, A - \lambda I \mid \vec{0}\,]$.
- L is diagonalizable $\Leftrightarrow$ V has an *eigenbasis* with respect to L (i.e., a basis for V consisiting of eigenvectors of L).
  - Not every linear transformation is diagonalizable—e.g., there is no eigenbasis of $\mathbb{R}^2$ for the endomorphism $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
  - Distinct eigenvalues give l.i. eigenvectors—so if L has n *distinct* eigenvalues, it must be diagonalizable.
- General strategy for diagonalizing a matrix A:
  - Find the eigenvalues of A. [If A is *upper-triangular*, these are simply its diagonal entries.]
  - Find l.i. eigenvectors: for each eigenvalue λ, find a basis for $N(A - \lambda I)$.
  - If n total eigenvectors result, they form an eigenbasis, and thus A is diagonalizable. [If not, A is not diagonalizable.]
    To diagonalize A:
    - form a change-of-basis matrix X with the eigenbasis as its columns, and
    - form a diagonal matrix Λ from the corresponding eigenvalues; then
    - $A = X\Lambda X^{-1}$.