Relationships between spanning sets, linearly independent sets, and bases

- $\mathscr{C}$ is linearly dependent in $V \iff$ one if its vectors is a linear combination of the rest.
- If $\mathscr{V}$ is linearly independent in V and $\mathscr{W}$ spans V, then $|\mathscr{V}| \leq |\mathscr{W}|$.
  - Method of proof: incrementally replace vectors of $\mathscr{V}$ into $\mathscr{W}$ (maintaining it as a spanning set) via careful use of the hypotheses and replacement rules.
  - Consequences:
    - If $\mathscr{C}_s$ spans V, $\mathscr{C}_{li}$ is linearly independent in V, and $\mathscr{C}_b$ is a basis for V, then $|\mathscr{C}_{li}| \leq |\mathscr{C}_b| \leq |\mathscr{C}_s|$.
    - All bases for V have the same size (because one is l.i. and the other spans V, and vice-versa).

Basis and dimension

- $\dim V$ is defined as the size of any basis for V.
  - This is intrinsic to V, because all bases for V have the same size.
- Any *linearly independent* collection in V can be *extended* to a basis for V.
  - Method of proof: iteratively insert a vector not in the collection's span, until the collection spans V.
  - Consequences:
    - A collection of size larger than $\dim V$ can't be linearly independent in V.
    - A linearly independent collection having size $\dim V$ *must be* a basis for V.
    - Every finite-dimensional vector space has a basis (extend the [l.i.] empty collection to a basis).
- Any *spanning set* for V can be *reduced* to a basis for V.
  - Method of proof: iteratively remove a vector that's a linear combination of the rest, until the collection is linearly independent in V.
  - Consequences:
    - A collection of size smaller than $\dim V$ can't span V.
    - A spanning set of V having size $\dim V$ *must be* a basis for V.

Bases and coordinate mappings

Suppose that $\mathscr{B} = (\, \vec{v}_1,\, \vec{v}_2,\, \ldots,\, \vec{v}_n \,)$ is an ordered basis for a vector space V.
We then have a corresponding **linear combination function** $[\mathscr{B}] : \mathbb{R}^n \to V$ and $\mathscr{B}$-**coordinate function** $[\mathscr{B}]^{-1} : V \to \mathbb{R}^n$.

- $[\mathscr{B}] : \mathbb{R}^n \to V$ maps each coordinate [coefficient] vector in $\mathbb{R}^n$ to the corresponding linear combination of $\mathscr{B}$ in V.
  - Because $[\mathscr{B}]$ is a basis for V, $[\mathscr{B}]$ pairs each column vector in $\mathbb{R}^n$ with one vector of V and vice-versa.
  - $[\mathscr{B}]$ is bijective and linear, so it gives a vector space *isomorphism* from $\mathbb{R}^n$ to V.
- $[\mathscr{B}]^{-1} : V \to \mathbb{R}^n$ is the inverse of the linear combination function $[\mathscr{B}]$.
  - $[\mathscr{B}]^{-1}$ maps each vector in $\vec{v} \in V$ to its $\mathscr{B}$-**coordinates** (the vector of coefficients that build $\vec{v}$ as a l.c. of $\mathscr{B}$).
  - $[\mathscr{B}]^{-1}$ is also bijective and linear, and thus also gives an isomorphism (from V to $\mathbb{R}^n$).
- Being inverse functions, $[\mathscr{B}][\mathscr{B}]^{-1} : V \to V$ and $[\mathscr{B}]^{-1}[\mathscr{B}] : \mathbb{R}^n \to \mathbb{R}^n$ are just the identity maps (i.e., these functions cancel each other).
- When dealing with bases, coordinates, and the functions $[\mathscr{B}]$ and $[\mathscr{B}]^{-1}$, always be alert to which vector space and basis you're dealing with and what the entries of a given column vector represent; with this kept in mind, all that these functions do is either form a linear combination of $\mathscr{B}$ or find the coefficients to build a vector as a linear combination of $\mathscr{B}$.

Common bases

- The **standard basis** for $\mathbb{R}^m$ is $(\, \vec{e}_1,\, \vec{e}_2,\, \ldots,\, \vec{e}_m \,)$, where $\vec{e}_j$ has all zero entries except for a 1 in the j^{th} position.
- For the subspace $P_n(x) = \{\, a_0 + a_1 x + \cdots + a_n x^n \;:\; a_0, a_1, \ldots, a_n \in \mathbb{R} \,\}$ of $\mathbb{R}[x]$, we have the basis $(\, 1,\, x,\, x^2,\, \ldots,\, x^n \,)$.

Vector space isomorphism and the FTVS

- An **isomorphism** from V to W is a bijection $\varphi : V \to W$ with the property of *linearity*:

$$\forall\, \vec{v}_1, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \ldots, \alpha_n,$$
$$\varphi(\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n) = \alpha_1\varphi(\vec{v}_1) + \cdots + \alpha_n\varphi(\vec{v}_n)$$

  - The *inverse* of an isomorphism is an isomorphism, and the *composition* of two isomorphisms is an isomorphism.
- Two vector spaces V and W are **isomorphic** if there exists an isomorphism from V to W.
  - For V and W to be isomorphic means that they, in a very literal sense, have the "same shape" as each other—they are mathematically equivalent as vector spaces.
  - An isomorphism (and its inverse) allow us to translate *any* linear algebra problem in V to one in W and vice-versa:
    - Each vector in V has a corresponding vector in W, and vice-versa.
    - Each linear combination in V has a corresponding linear combination in W, and vice-versa.
    - Each spanning set, linearly independent set, or basis in V has a corresponding spanning set, linearly independent set, or basis in W, and vice-versa.
    - Any *other* problem formed from the concept of linear combination can be translated from V to W, and vice-versa.
- The **FTVS**: Any n-dimensional vector space V [over $\mathbb{R}$] is isomorphic to $\mathbb{R}^n$.
  - Method of proof: Taking any *basis* $\mathscr{B} = (\,\vec{v}_1, \ldots, \vec{v}_n\,)$ for V gives us a coordinate isomorphism $[\mathscr{B}]^{-1} : V \to \mathbb{R}^n$.
- Corollary to the FTVS: Any two vector spaces V and W of the same dimensions [over the same field] are isomorphic.
  - Method of proof: Take bases for V and W, giving coordinate isomorphisms for each; invert one and compose to obtain an isomorphism from V to W.
  - Consequence: dimension is the *fundamental* property intrinsic to an abstract vector space (considered up to isomorphism).
- Computation via the FTVS: We can translate any linear algebra problem in any finite-dimensional vector space into a column vector problem, simply by choosing a basis for the vector space and working with coordinates (then translating our result).

Column-vector methods

Suppose that we have collections of ***column vectors*** $\mathscr{C}, \mathscr{D}$ in $\mathbb{R}^m$.

Spanning

- $\vec{v} \in \operatorname{span}\mathscr{C} \iff [\,\mathscr{C} \mid \vec{v}\,]$ is consistent. $\qquad \vec{v} \notin \operatorname{span}\mathscr{C} \iff [\,\mathscr{C} \mid \vec{v}\,]$ is inconsistent.
- $\operatorname{span}\mathscr{C} \supset \operatorname{span}\mathscr{D} \iff [\,\mathscr{C} \mid \mathscr{D}\,]$ is consistent. $\qquad \operatorname{span}\mathscr{C} = \operatorname{span}\mathscr{D} \iff [\,\mathscr{C} \mid \mathscr{D}\,]$ and $[\,\mathscr{D} \mid \mathscr{C}\,]$ and are both consistent.
- $\mathscr{C}$ spans $\mathbb{R}^m \iff [\,\mathscr{C}\,]$ gives a pivot in every row.

Linear Independence

- $\mathscr{C}$ is linearly independent $\iff [\,\mathscr{C}\,]$ gives a pivot in every column. $\qquad \mathscr{C}$ is linearly dependent if not.
- Coefficients for linear relations on $\mathscr{C}$ are found by solving $[\,\mathscr{C} \mid \vec{0}\,]$. (for a *nontrivial* linear rel'n, set any free variable to 1)

Bases

- $\mathscr{C}$ is a basis for $\mathbb{R}^m \iff [\,\mathscr{C}\,]$ gives a pivot in every row and column.
- To find a basis for $\operatorname{span}\mathscr{C}$, take the vectors of $\mathscr{C}$ that give pivots in $[\,\mathscr{C}\,]$.
- To extend a l.i. collection $\mathscr{C}$ to a basis for $\mathbb{R}^m$, append the standard basis to $\mathscr{C}$ and keep the ones that give pivots.

Coordinates

Suppose that $\mathscr{B}$ is a basis for $\mathbb{R}^m$.

- Compute $[\mathscr{B}]\vec{x}$ as usual, simply by using the entries of $\vec{x}$ to form a linear combination of $\mathscr{B}$.
- Compute $[\mathscr{B}]^{-1}\vec{x}$ by solving $[\,\mathscr{B} \mid \vec{x}\,]$ and expressing the solution as a column vector.